Morse theory

on the moduli space of Higgs bundles

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Definitions

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Not a "good" quotient.
Restricting to the reductive representations provides a "good" moduli space:

Definition

Character Variety: $\mathcal{R}(G) = \mathrm{Hom}^{+}(\pi_1(\Sigma), G)/\mathrm{G}$

This is an analytic variety.

Importance:

- The spaces $\mathcal{R}(G)$ arise in a variety of contexts from geometric structures to applications in low dimensional topology.
- How does $\mathrm{Mod}(\Sigma) \curvearrowright \mathcal{R}(G)$.
- Components of $\mathcal{R}(G)$ of special geometric significance.
- It is essential to study its geometry and topology.

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The most primitive topological property of $\mathcal{R}(G)$ is the number of its connected components.

- For G complex or compact: the connected components of this moduli space can be easily described (results by Narasimhan-Seshadri, Ramanathan, Hitchin).
- For G non-compact real form of semisimple, complex Lie group: more complicated.

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This allows us to define characteristic classes for representations

$$c(\rho) := c(E_\rho)$$

and

$$\mathcal{R}_d(G) = \{\rho : c(\rho) = d\}$$

is closed subset of $\mathcal{R}(G)$.

This provides a partition of the character variety into closed, disjoint subspaces, but not necessarily into connected components.

Moreover, the holonomy representation provides

$$\mathcal{M}_{flat}(G) \simeq \mathcal{R}(G)$$

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The Narasimhan-Seshadri Theorem

Equipping the surface with a complex structure $J \in \mathcal{T}(\Sigma)$ transforms our topological surface Σ into a Riemann surface

$$X = (\Sigma, J)$$

and opens the way for the application of holomorphic techniques and the theory of Higgs bundles.

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and so correspond to $\{\rho : \pi_1(X) \to \mathrm{U}(n)\}$.

Moduli space of Higgs bundles

The holomorphic objects that correspond to representations $\{\rho : \pi_1(X) \to G\}$, which are not necessarily unitary are the Higgs bundles.These first appeared into the study of self-dual Yang-Mills equations over Riemann surfaces by N. Hitchin in 1987.

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Definition

Let K be the canonical line bundle over X. An SL$(n, \mathbb{C})$-Higgs bundle is a pair (E, Φ) where:

- $E \to X$ is a holomorphic vector bundle
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similarly for any real, semisimple, real group G.
Introducing a suitable stability condition for these pairs provides the construction of a "good" moduli space under the action of the complex gauge group $\mathcal{G}$:

$$\mathcal{M}_{Higgs}(G)$$

The non-abelian Hodge correspondence 1/3

In the fundamental work of N. Hitchin and C. Simpson this condition is connected to the existence of solutions to particular equations in gauge theory:

Theorem

If (E, Φ) is stable then and only then there exists a hermitian metric h on E such that $\left(\bar{\partial}_A, \Phi\right)$ solves the Hitchin equations

$$\begin{cases} F_A + [\Phi, \Phi^*] = 0 \\ \bar{\partial}_A \Phi = 0 \end{cases}$$

where $\nabla_A = \nabla\left(\bar{\partial}_A, h\right)$ is the Chern connection.

The non-abelian Hodge correspondence 2/3

Notice that if for a choice of a hermitian metric h the pair $\left(\bar{\partial}_A, \Phi\right)$ satisfies the Hitchin equations then

$$\nabla = \nabla_A + \Phi + \Phi^*$$

is flat.

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is flat.
And a Theorem of S. Donaldson and K. Corlette provides

Theorem

A flat G-connection comes from a solution to the Hitchin equations if and only if the holonomy of this connection is reductive.

The non-abelian Hodge correspondence 3/3

Putting together the main theorems we discussed so far:

Theorem (Non-abelian Hodge correspondence)

$$\begin{array}{ccccccc} \mathcal{M}_{Higgs}(G) & \simeq & \mathcal{M}_{Hitchin}(G) & \simeq & \mathcal{M}_{flat}(G) & \simeq & \mathcal{R}(G) \\ (E,\Phi) & & \left(\bar{\partial}_A,\Phi\right) & & \nabla = \nabla_A + \Phi + \Phi^* & & \rho : \pi_1(X) \to G \\ \textit{stable} & & \textit{solves the H. eq.} & & \textit{flat} & & \textit{reductive} \end{array}$$

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- Historical review
- Different mathematical objects describe essentially the same moduli space

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- Historical review
- Different mathematical objects describe essentially the same moduli space
- The extremes are most interesting; Very hard to pair up individual objects
- Allows us to use techniques from other disciplines

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Introducing Morse theoretic techniques

- Morse theoretic techniques have been very effective into the study of moduli of vector bundles [Atiyah-Bott]. The use of Morse theory into $\mathcal{M}_{Hitchin}(\mathsf{SL}(n,\mathbb{C}))$ was introduced by N. Hitchin in his articles:
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- These techniques provide a relationship between the critical points of a real valued function (to be called a Morse function) on a smooth manifold $\mathcal{M}$ and the topology of $\mathcal{M}$.
- We need to equip our moduli space with a symplectic structure and define an appropriate Morse function on it. For this, focus on $\mathcal{M}_{Hitchin}(G)$ from the gauge theory point of view.

$\mathcal{M}_{Hitchin}(G)$ has a symplectic structure 1/2

Recall that $\mathcal{M}_{Hitchin}(G)$ is the space of solutions $\left(\bar{\partial}_A, \Phi\right)$ of the Hitchin equations

$$\begin{cases} F_A + [\Phi, \Phi^*] = 0 \\ \bar{\partial}_A \Phi = 0 \end{cases}$$

The space

$$\mathcal{A} \times \Omega = \mathcal{A}\left(X, \mathsf{ad}P\right) \times \Omega^{1,0}\left(X, \mathsf{ad}P \otimes \mathbb{C}\right)$$

has a natural Kähler metric defined by

$$g\left(\left(A, \Phi\right), \left(A, \Phi\right)\right) = 2i \int_X \mathsf{Tr}\left(A^* A + \Phi\Phi^*\right)$$

and the gauge group $\mathcal{G}$ acts on $\mathcal{A} \times \Omega$ by conjugation.

$\mathcal{M}_{Hitchin}(G)$ has a symplectic structure 2/2

The tangent space to $\mathcal{A}$ carries the Kähler metric $g(\Psi, \Psi) = 2i \int\limits_X \mathrm{Tr}(\Psi^* \Psi)$ and the moment map for the action of $\mathcal{G}$ on $\mathcal{A}$ is then $\mu_1(A) = F_A$.
Considering the natural action of $\mathcal{G}$ on $\Omega^{1,0}(\mathrm{ad}P \otimes \mathbb{C})$ with Kähler metric $g(\Phi, \Phi) = 2i \int\limits_X \mathrm{Tr}(\Phi \Phi^*)$ the moment map is $\mu_2(\Phi) = [\Phi, \Phi^*]$.

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Therefore the equation $F_A + [\Phi, \Phi^*] = 0$ can be interpreted as the zero set of the moment map

$$\mu(A, \Phi) = \mu_1(A, \Phi) + \mu_2(A, \Phi)$$

for the action of $\mathcal{G}$ on $\mathcal{A} \times \Omega$.

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The Hitchin functional 1/2

The key observation:
There is an action of $\mathbb{C}^*$ on the space of solutions of the Hitchin equations

$$\begin{cases} F_A + [\Phi, \Phi^*] = 0 \\ \bar{\partial}_A \Phi = 0 \end{cases}$$

by scaling the Higgs field

$$\psi : \mathbb{C}^* \times \mathcal{M} \to \mathcal{M}$$
$$\left(\lambda e^{i\theta}, \left(\bar{\partial}_A, \Phi\right)\right) \mapsto \left(\bar{\partial}_A, \lambda e^{i\theta}\Phi\right)$$

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This restricts to a Hamiltonian action of the circle S^1 on $\mathcal{M}$ away from its singular locus and we may calculate the moment map for this action to be

$$\mu\left(\bar{\partial}_A, \Phi\right) = -\frac{1}{2}\|\Phi\|^2 = -i\int_X \mathrm{Tr}\left(\Phi\Phi^*\right)$$

The Hitchin functional 2/2

We choose to use the positive function instead (Hitchin functional):

$$\begin{aligned} f : \mathcal{M} &\to \mathbb{R} \\ (\bar{\partial}_A, \Phi) &\mapsto \|\Phi\|^2 \end{aligned}$$

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The following is a consequence of Uhlenbeck's weak compactness theorem

Theorem (N. Hitchin)

The Hitchin functional is a proper Morse-Bott function.

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Theorem (N. Hitchin)

The Hitchin functional is a proper Morse-Bott function.

Moreover, the critical points of f are exactly the fixed points of the S^1-action, since

$$df = -2i\,(X)\,\omega$$

for X a vector field generating the circle action.

Counting components 1/2

In general:

Proposition

Let Z be Hausdorff and $f : Z \to \mathbb{R}$ proper, bounded below. Then f attains a minimum on each connected component of Z. Furthermore, if the space of local minima of f is connected then so is Z.

Therefore:

- We need to identify the local minima of the Hitchin functional f.
- The G-Higgs bundles corresponding to fixed points of the circle action are called Hodge bundles. These can be described explicitly for the different types of classical Lie groups G.
- Let $\mathcal{M} \subseteq \mathcal{M}_{Hitchin}(G)$ closed subspace and $\mathcal{N} \subseteq \mathcal{M}$ the subspace of local minima of f on $\mathcal{M}$. If $\mathcal{N}$ is connected, then so is $\mathcal{M}$.

Counting components 2/2

A few examples..

$\mathcal{R}(G)$	number of components
$\mathcal{R}(\mathsf{PSL}(n,\mathbb{R}))$, for $n > 2$	3 if n is odd, 6 if n is even
$\mathcal{R}(\mathsf{U(p,q)})$	$2(p+q)\min(p,q)(g-1)+\gcd(p,q)$
$\mathcal{R}(\mathsf{Sp}(4,\mathbb{R}))$	$6 \cdot 2^{2g} + 10g - 13$
$\mathcal{R}(\mathsf{Sp}(2n,\mathbb{R}))$	$6 \cdot 2^{2g} + 4g - 5$

Thαnk you!