Puzzles, triangulations and moduli spaces

May 25th, 2016, CAaGiG, Anogeia

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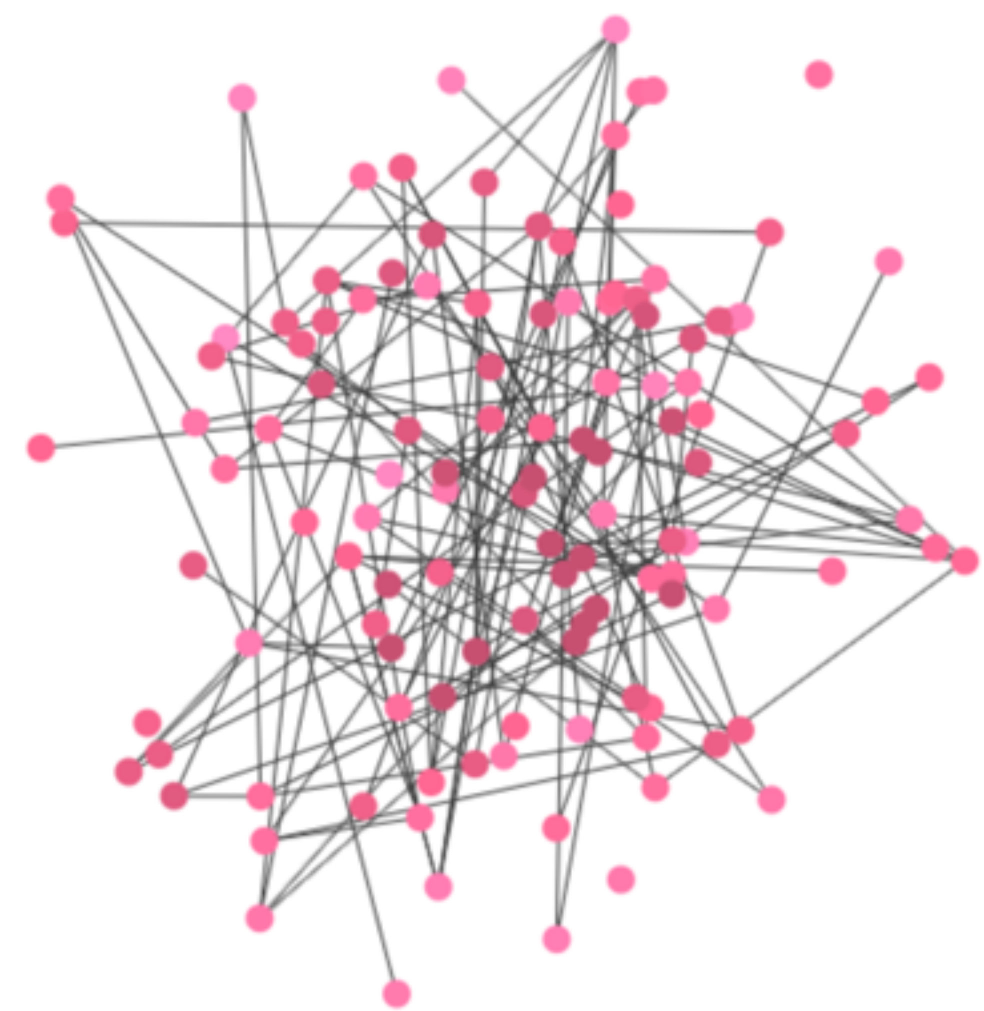
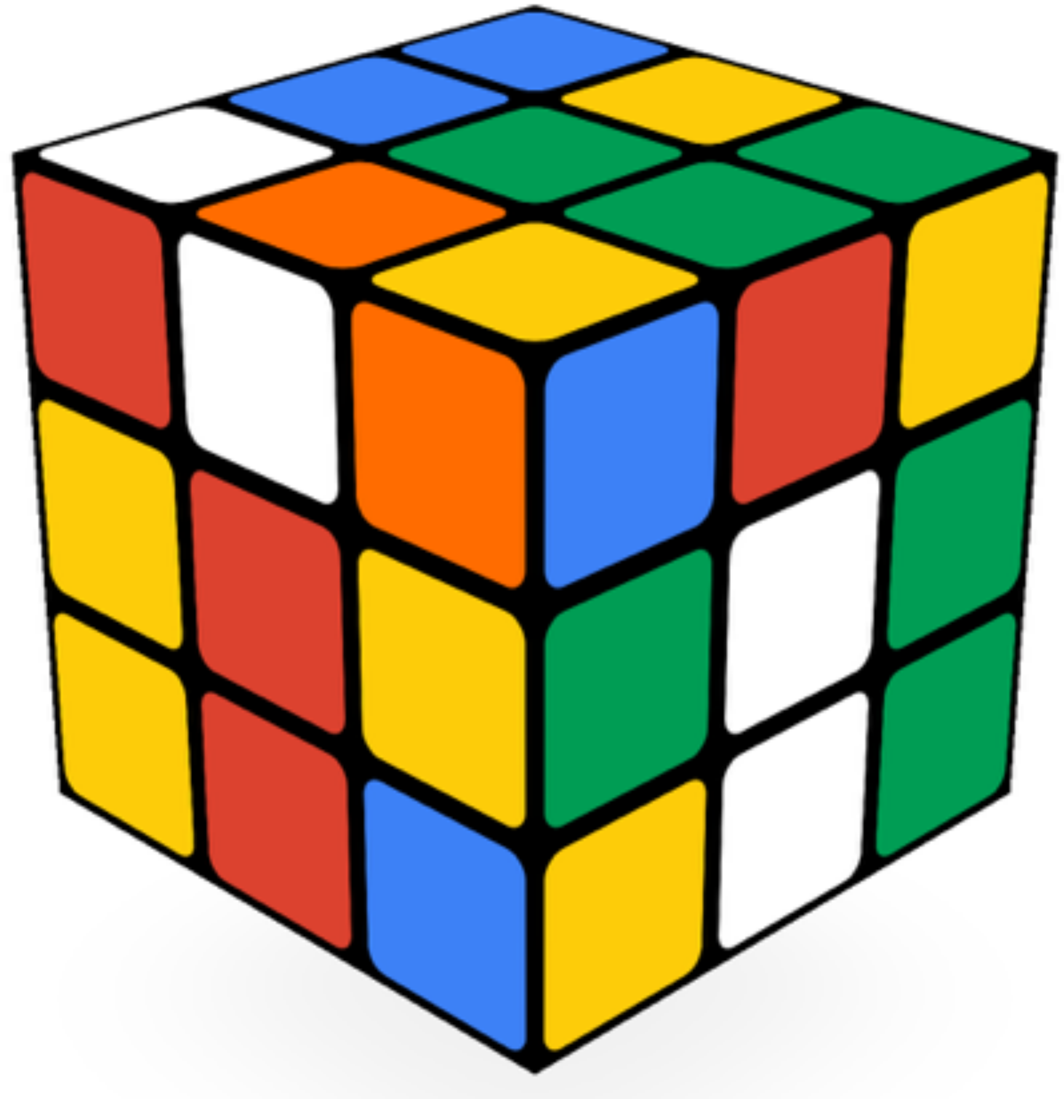
Université de Fribourg

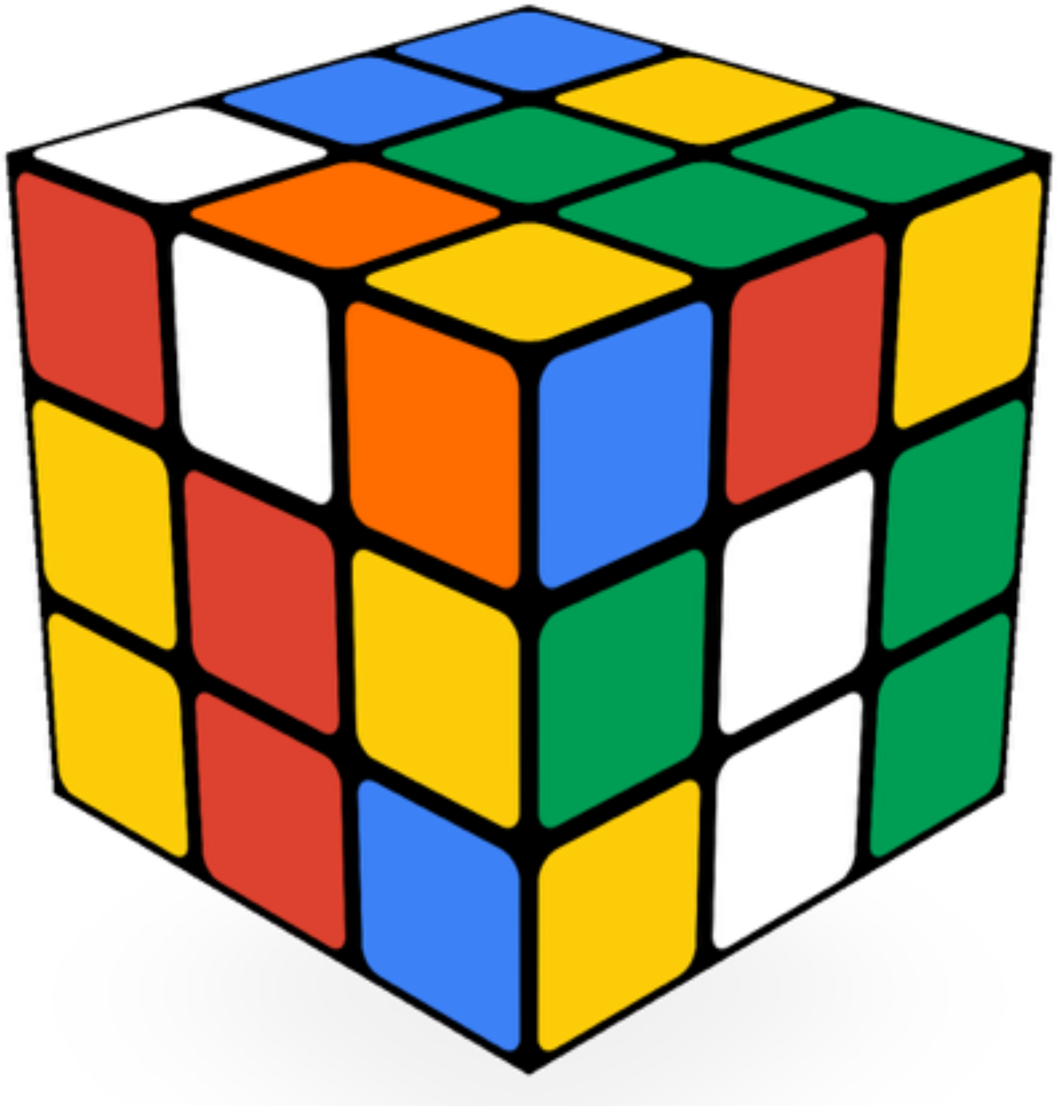




Rubik’s cube graph:

Vertices = configurations

Edges = elementary moves (face rotations)



Rubik's cube graph:

<u>Goal:</u> understand the shape / size / topology of this configuration space.

Size:

There are roughly 43 10^18 different configurations.

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Corollary:
Humanity has never seen all configurations.

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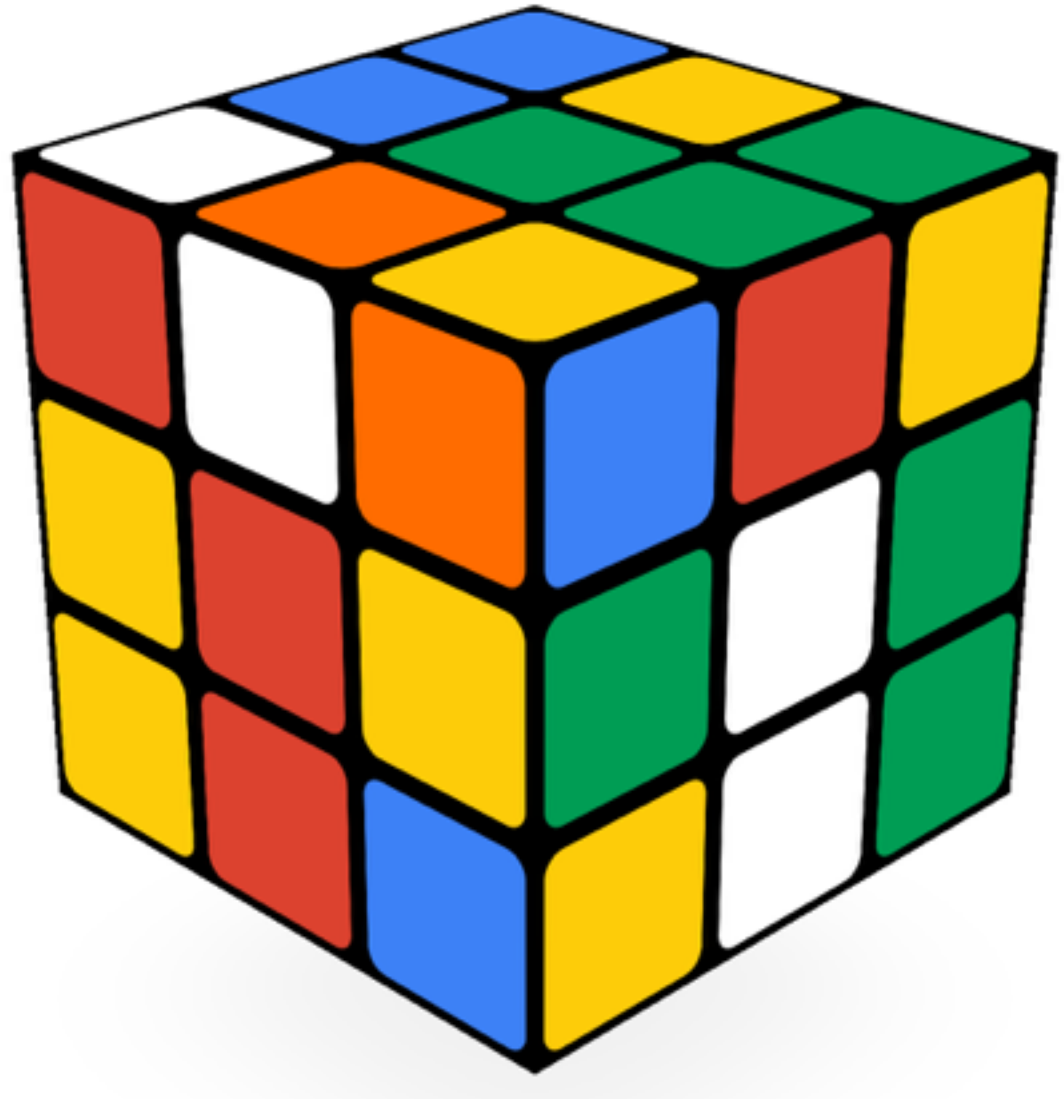
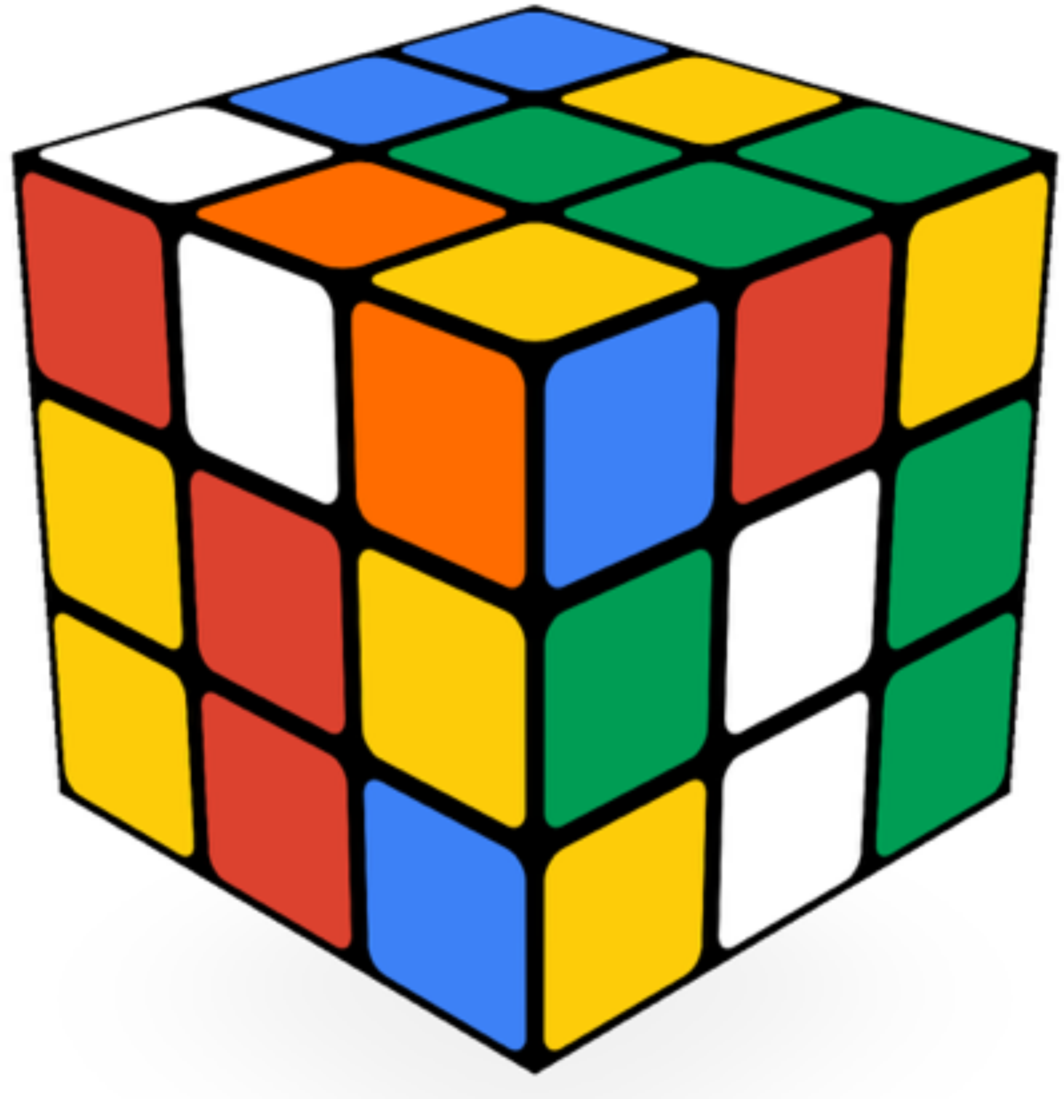
Proof:
Rubik's cube is 42 years old.
There are roughly 350 million cubes, less than 10^9.
Roughly 10^4 moves an hour.
Roughly 10^4 hours a year.
So we've certainly seen at most 42 10^17 configurations.



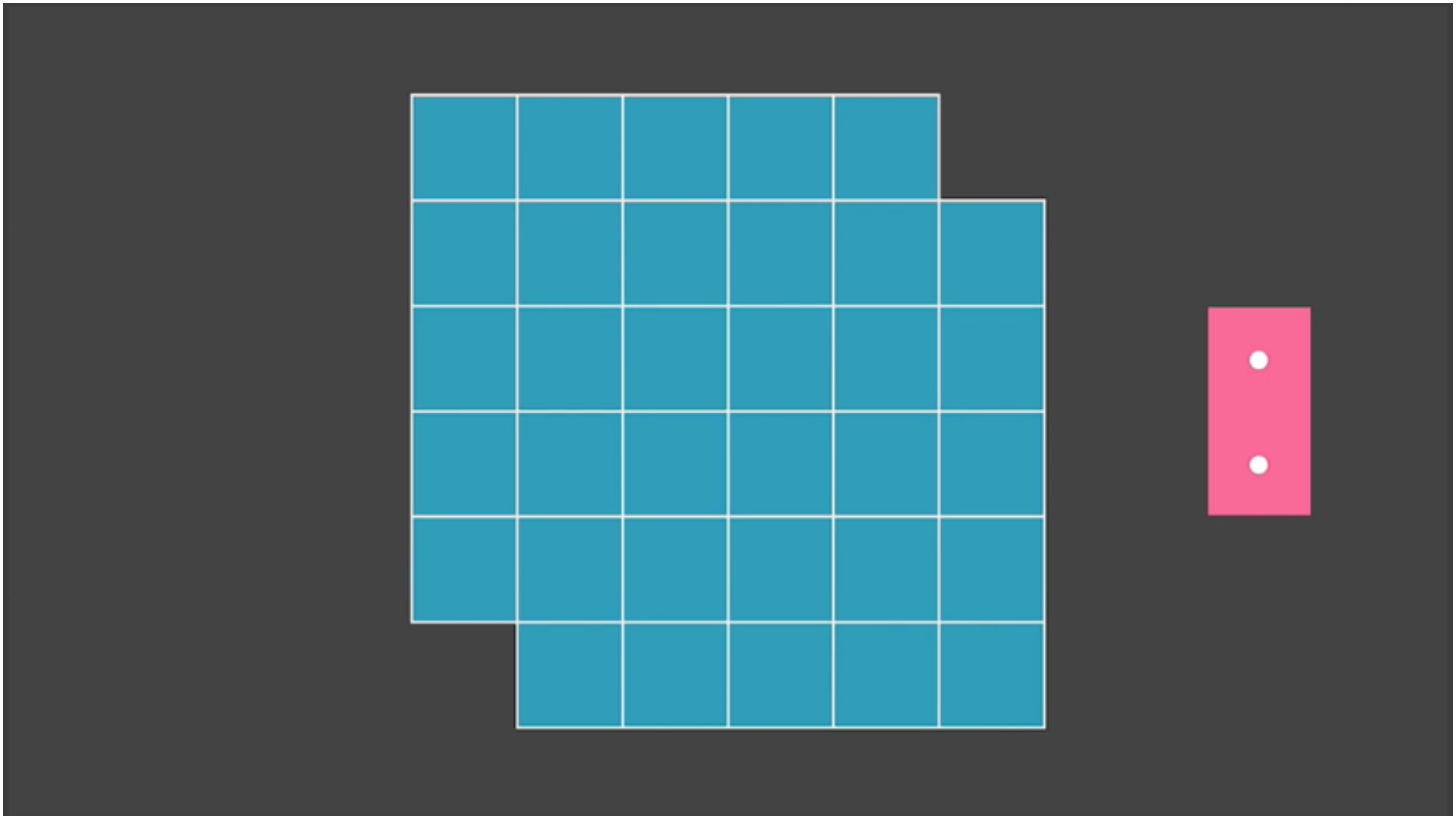
Theorem (Rokicki-Kociemba-Davidson-Dethridge, 2011)

The diameter of the Rubik graph is 20.

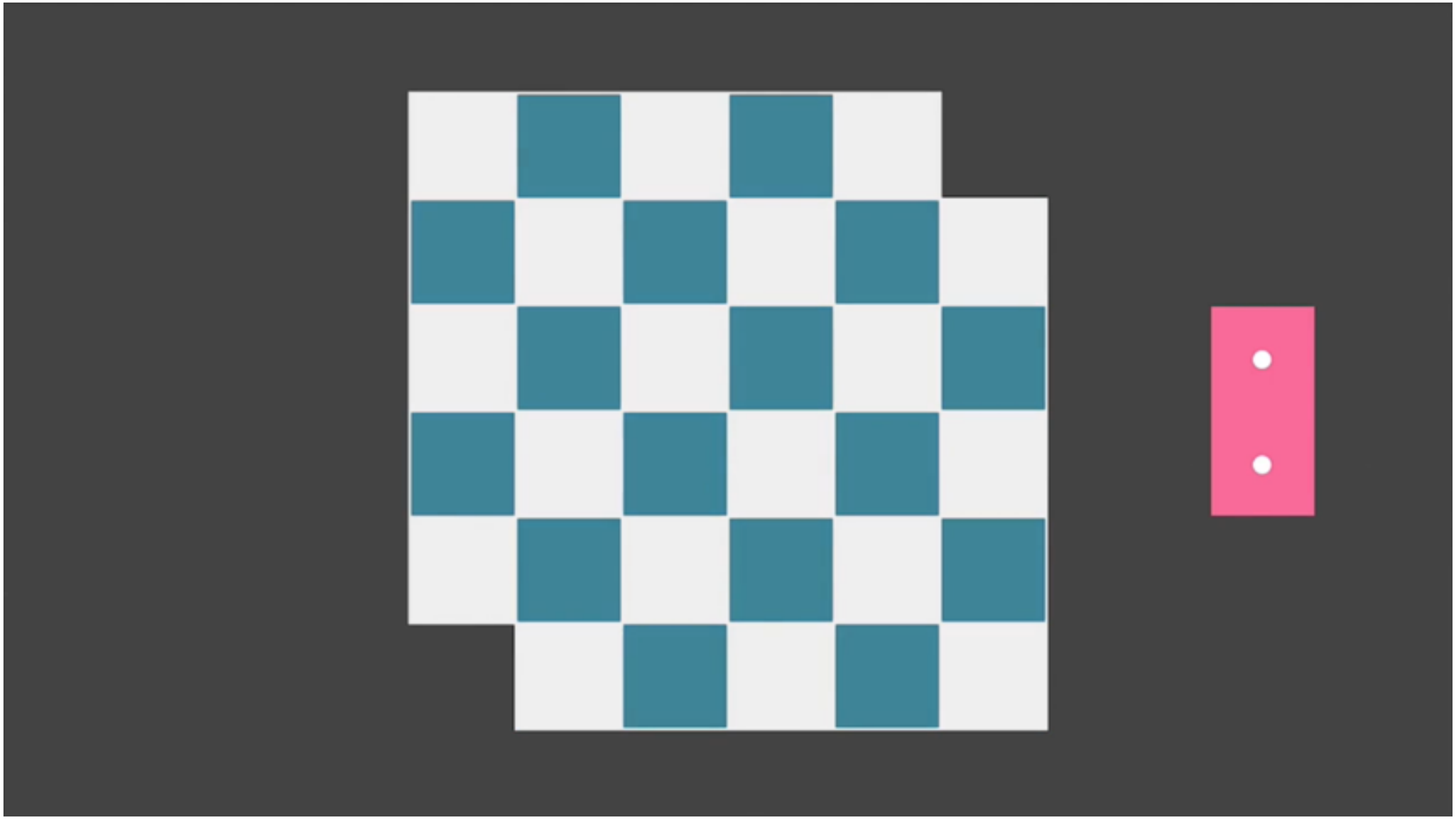


Theorem (Rokicki-Davidson, 2015)

The diameter of the Rubik graph with the *quarter-turn* metric is 26.



The board cannot be covered.



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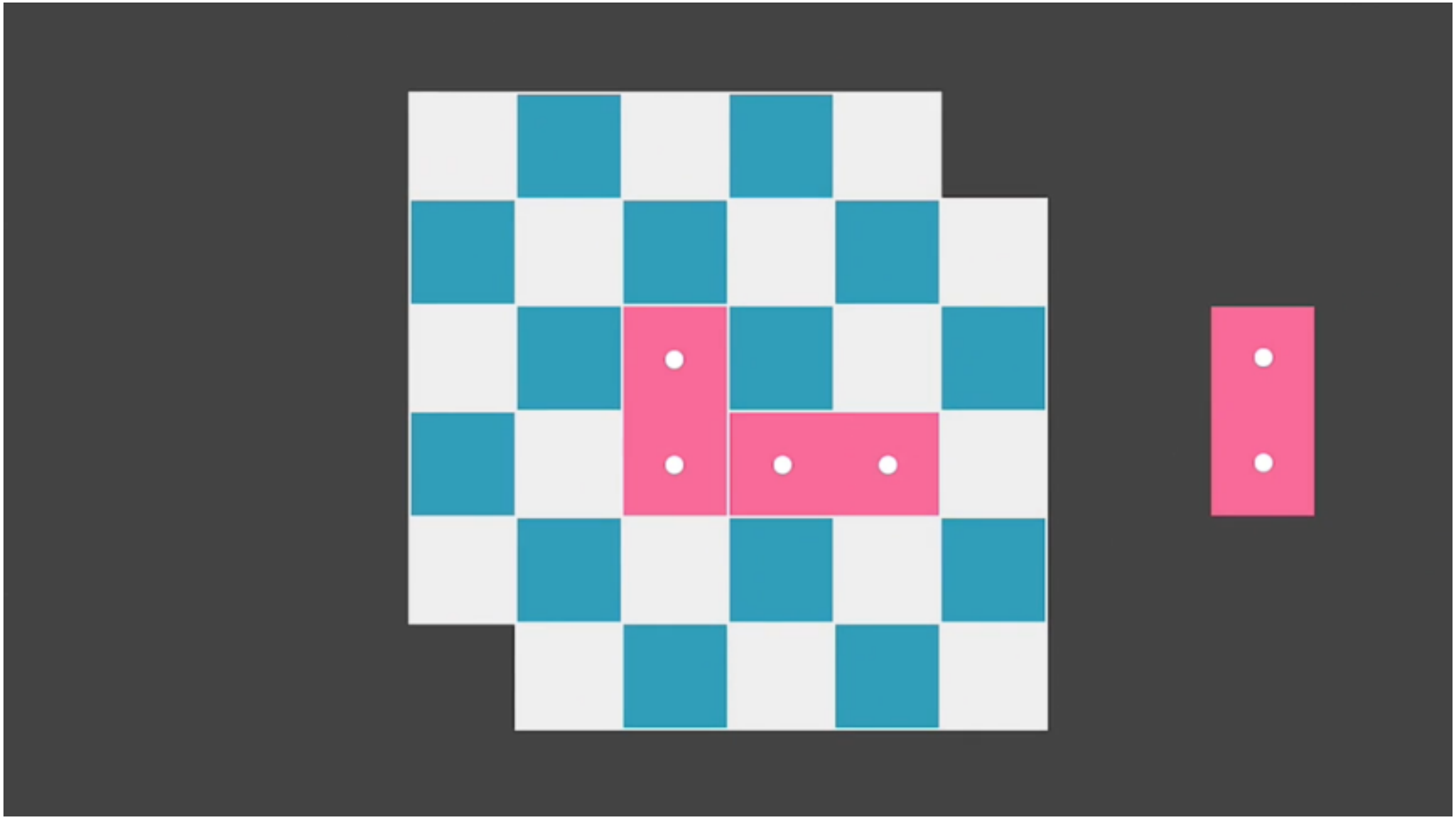

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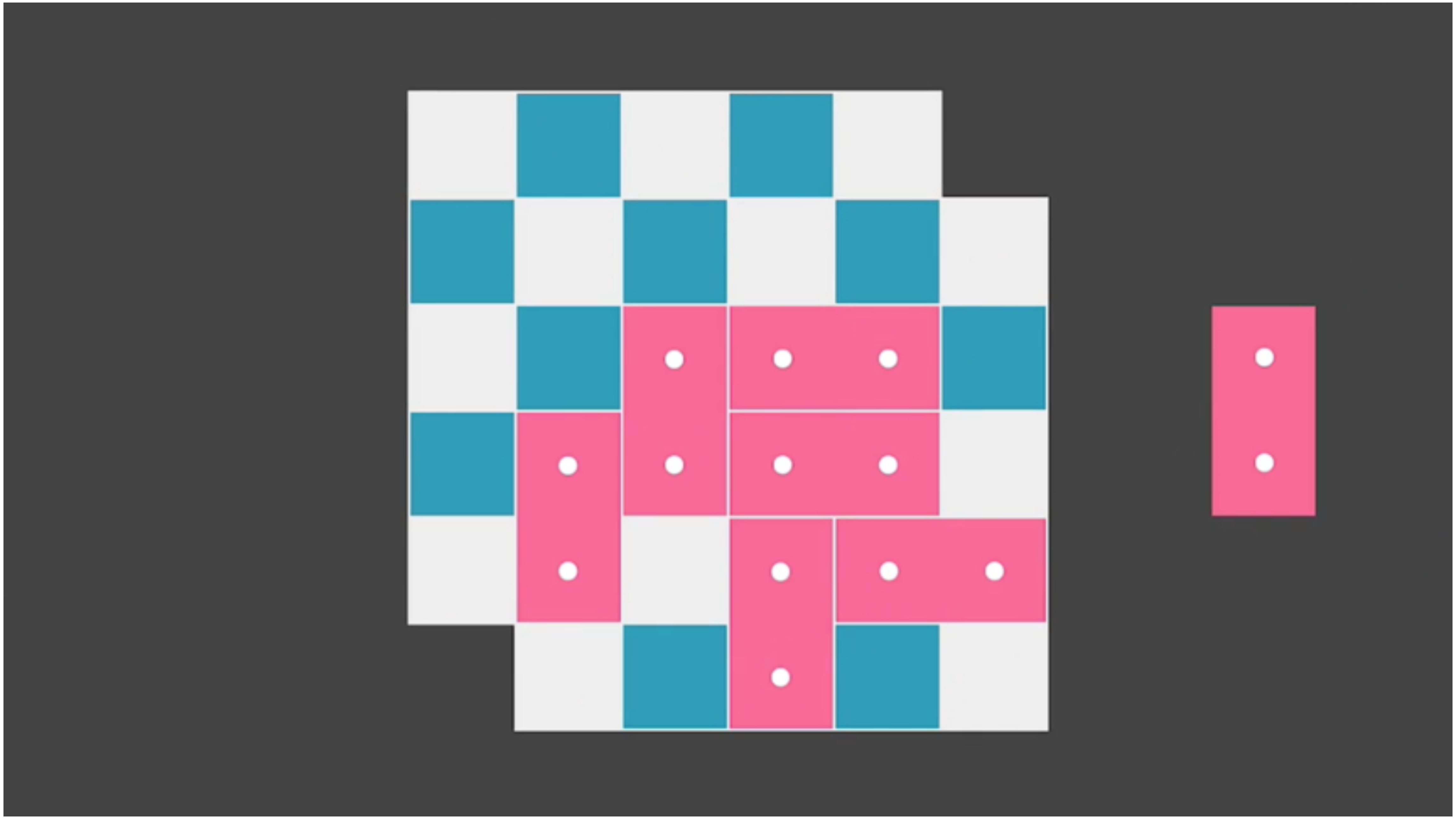

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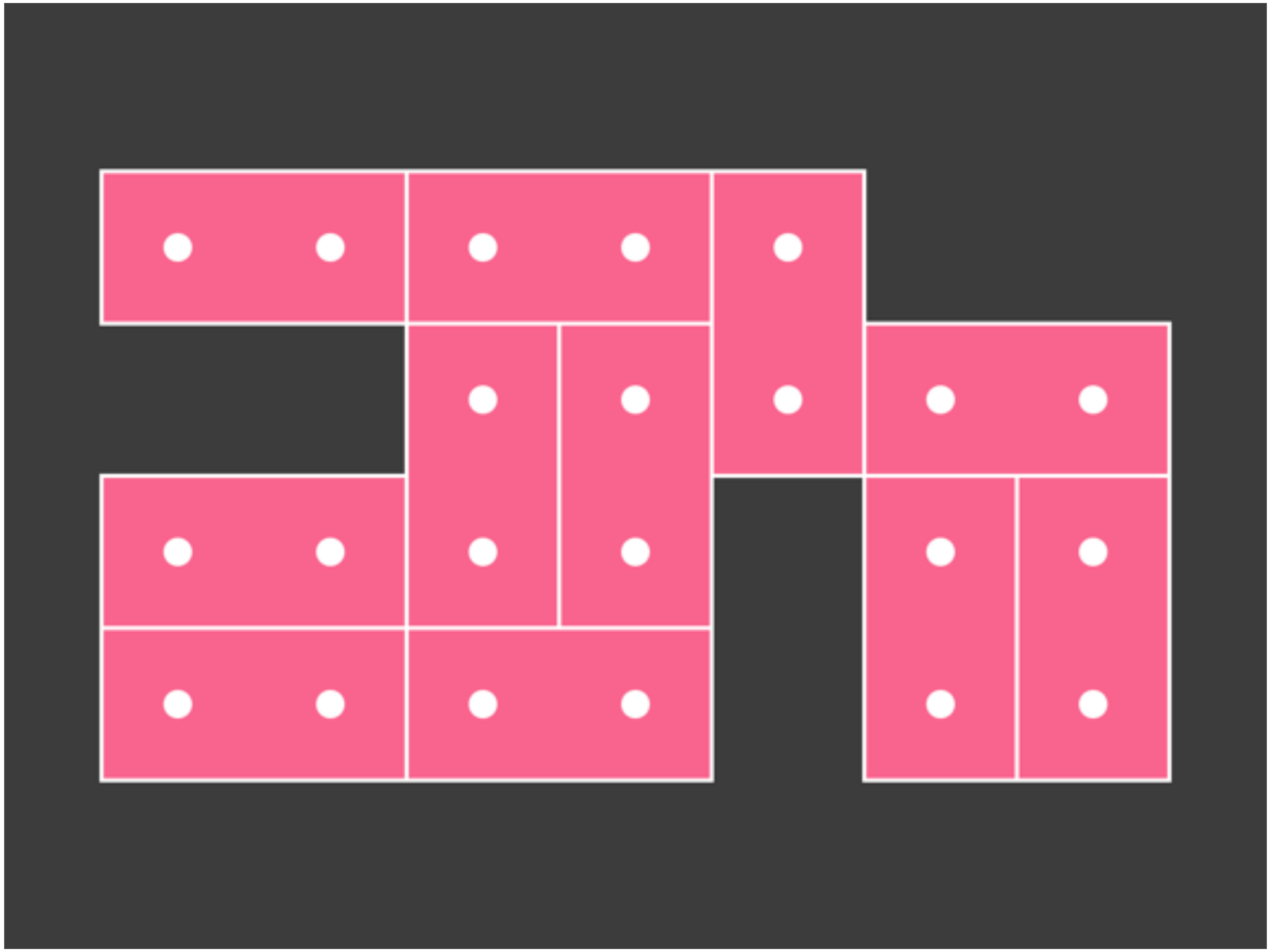
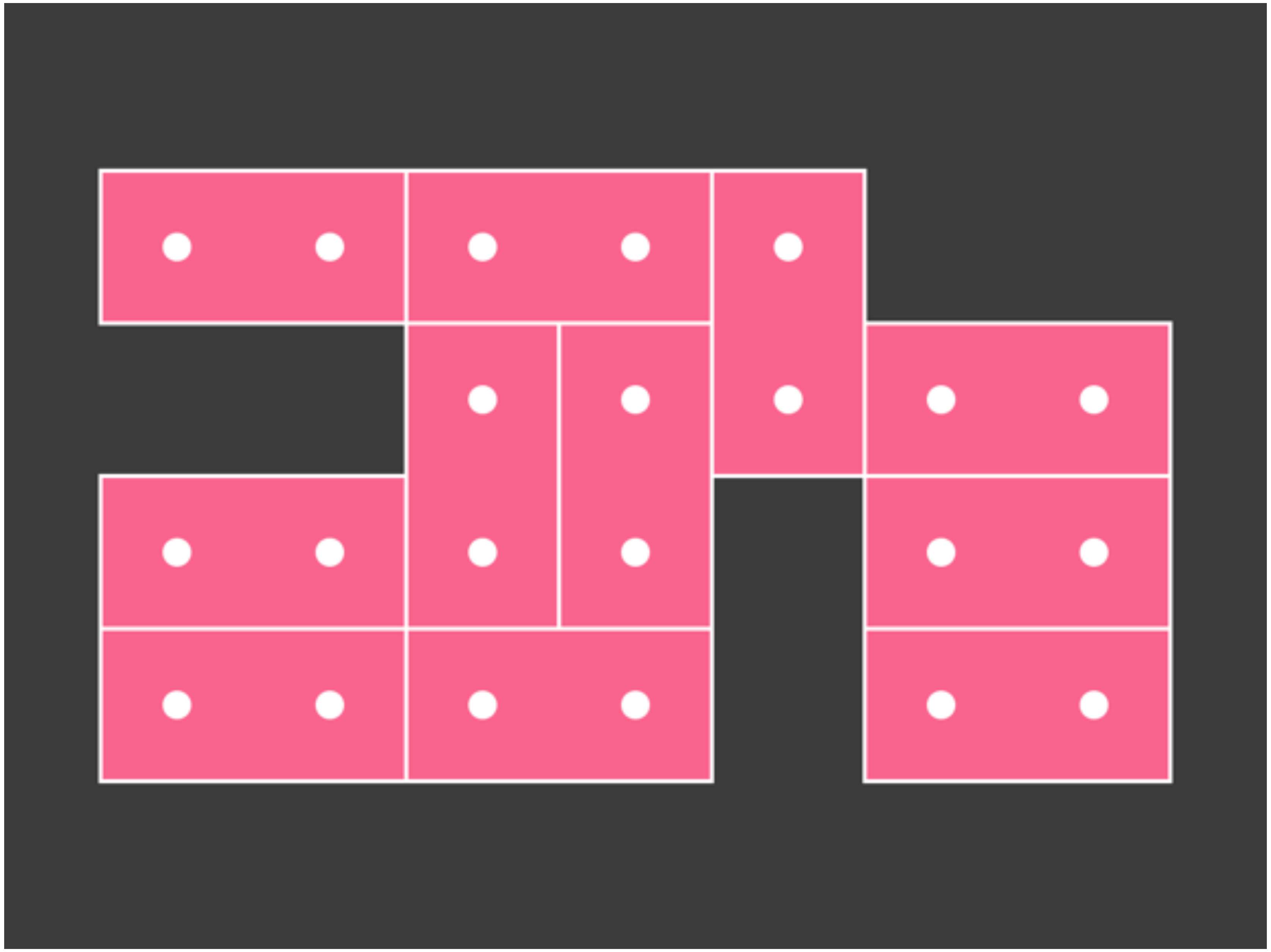


Flip graph:

Flip two side by side dominos to get a new tiling.



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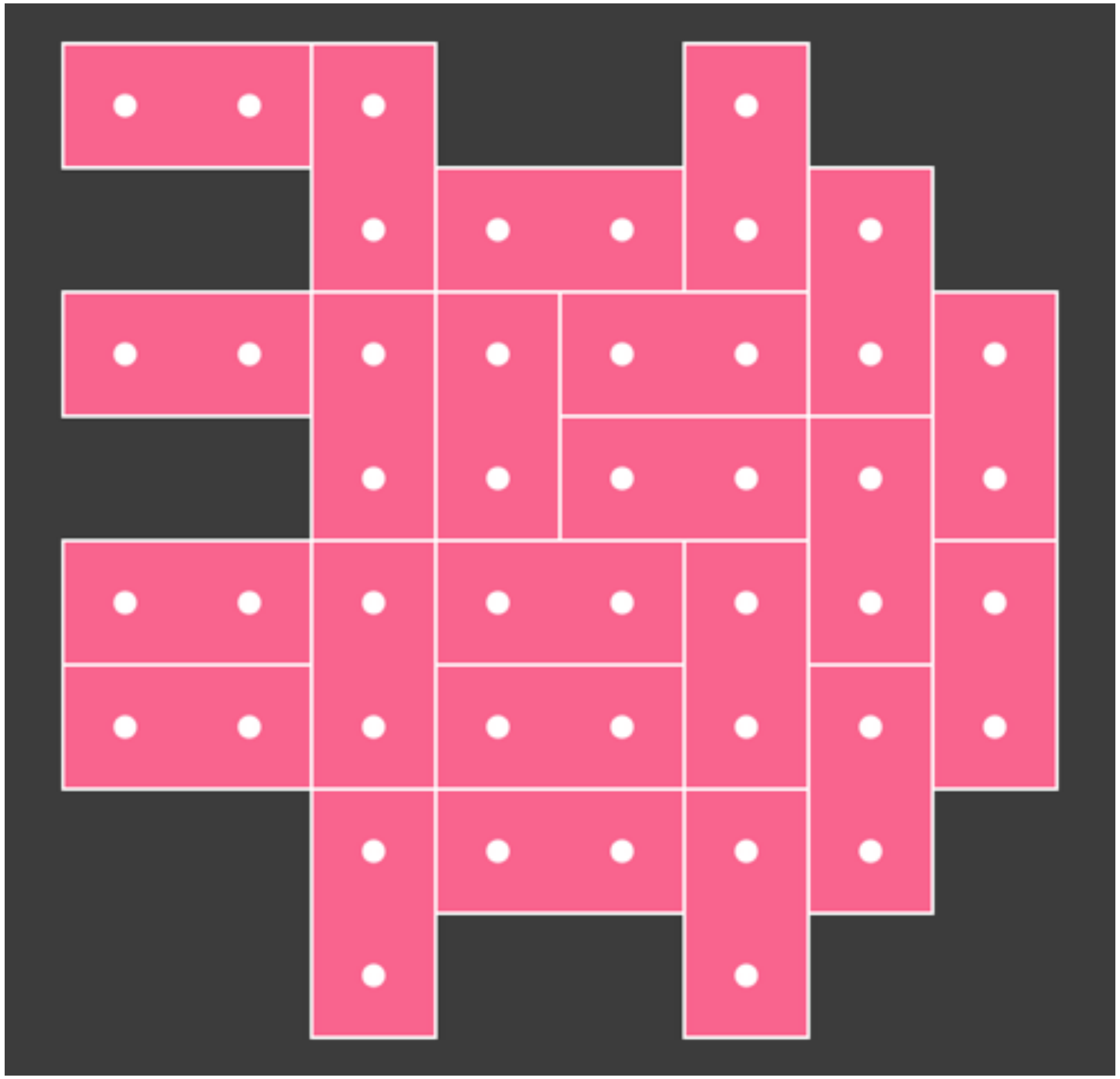
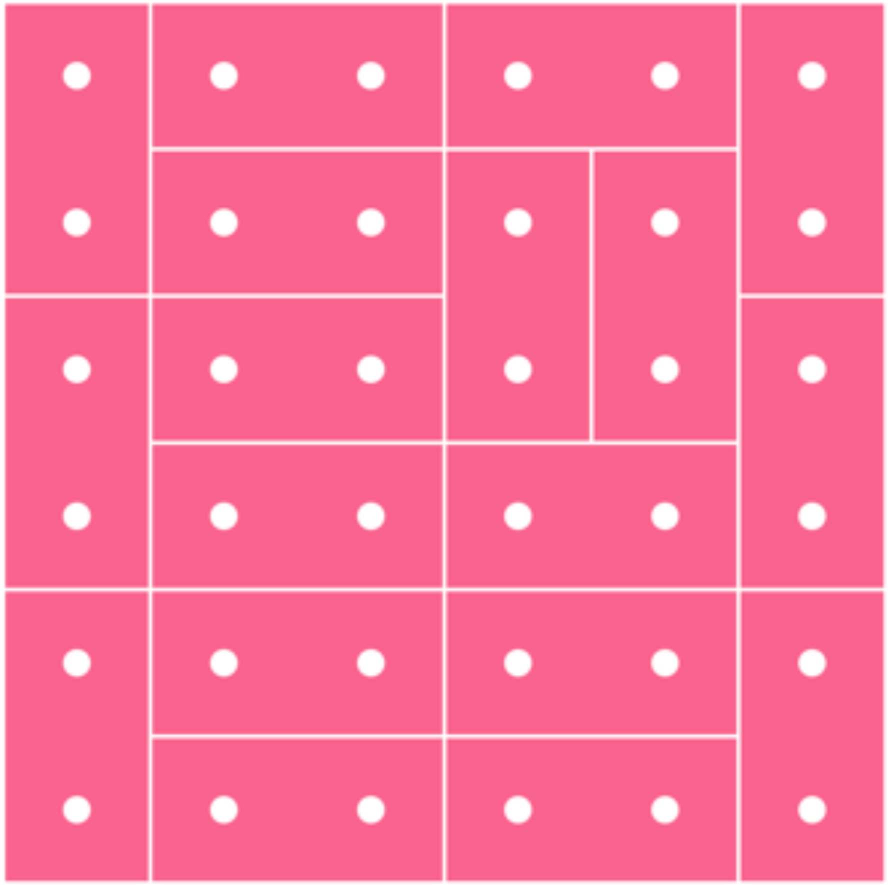


<u>Theorem</u> (Thurston, Elkies-Kuperberg-Propp)

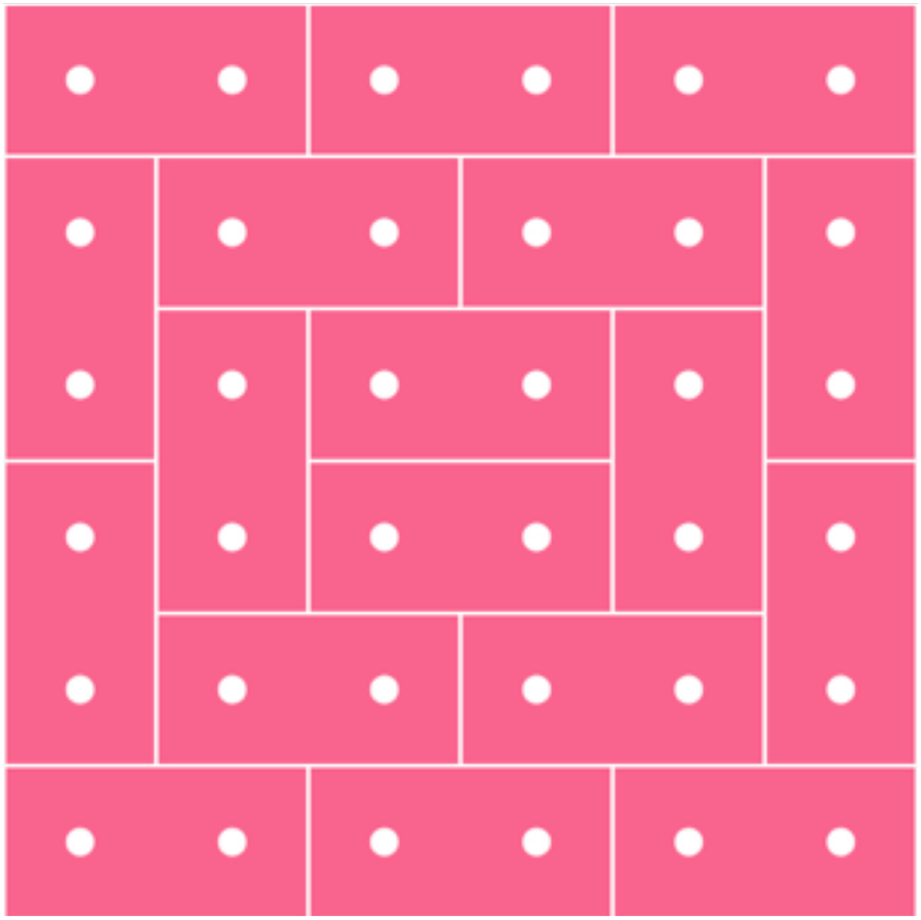
The flip graph of a simply connected shape is connected.

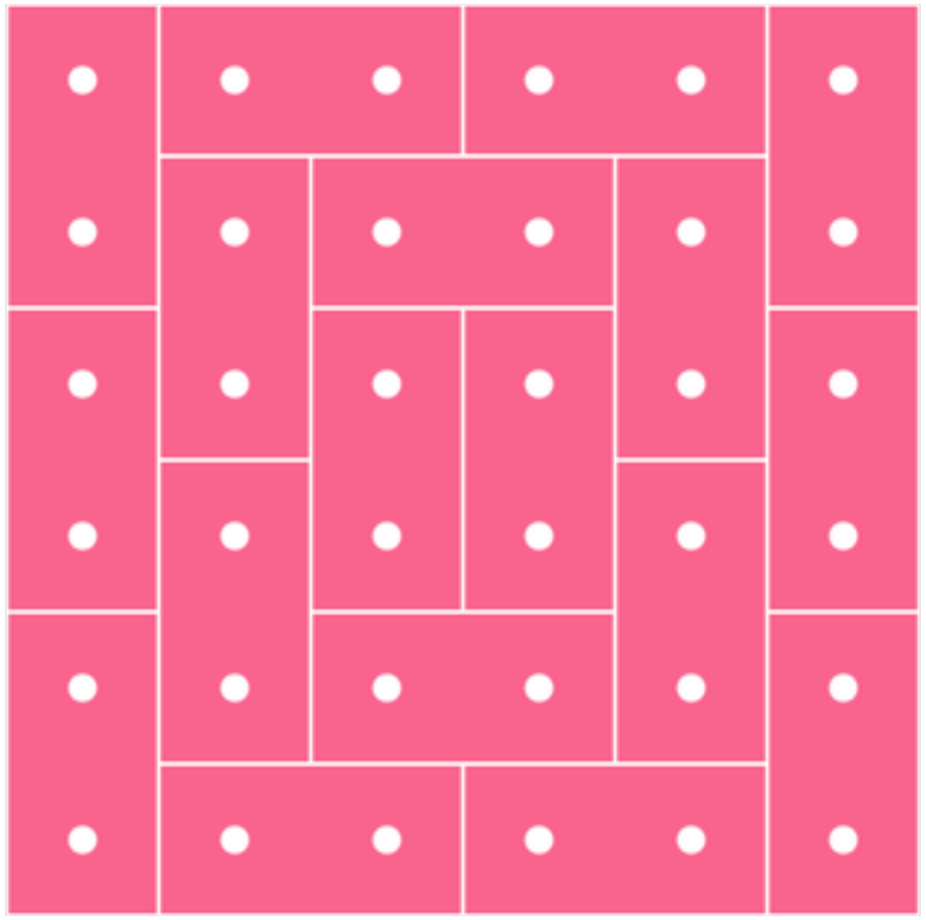


<u>Theorem</u> (Temperly-Fischer, Kasteleyn, 1961)

The number of tilings of an n by n square (n even) is

$$\Pi_{j,k=1}^{\frac{n}{2}} \left(4\cos^2\left(\frac{\pi j}{n+1}\right) + 4\cos^2\left(\frac{\pi k}{n+1}\right)\right)$$





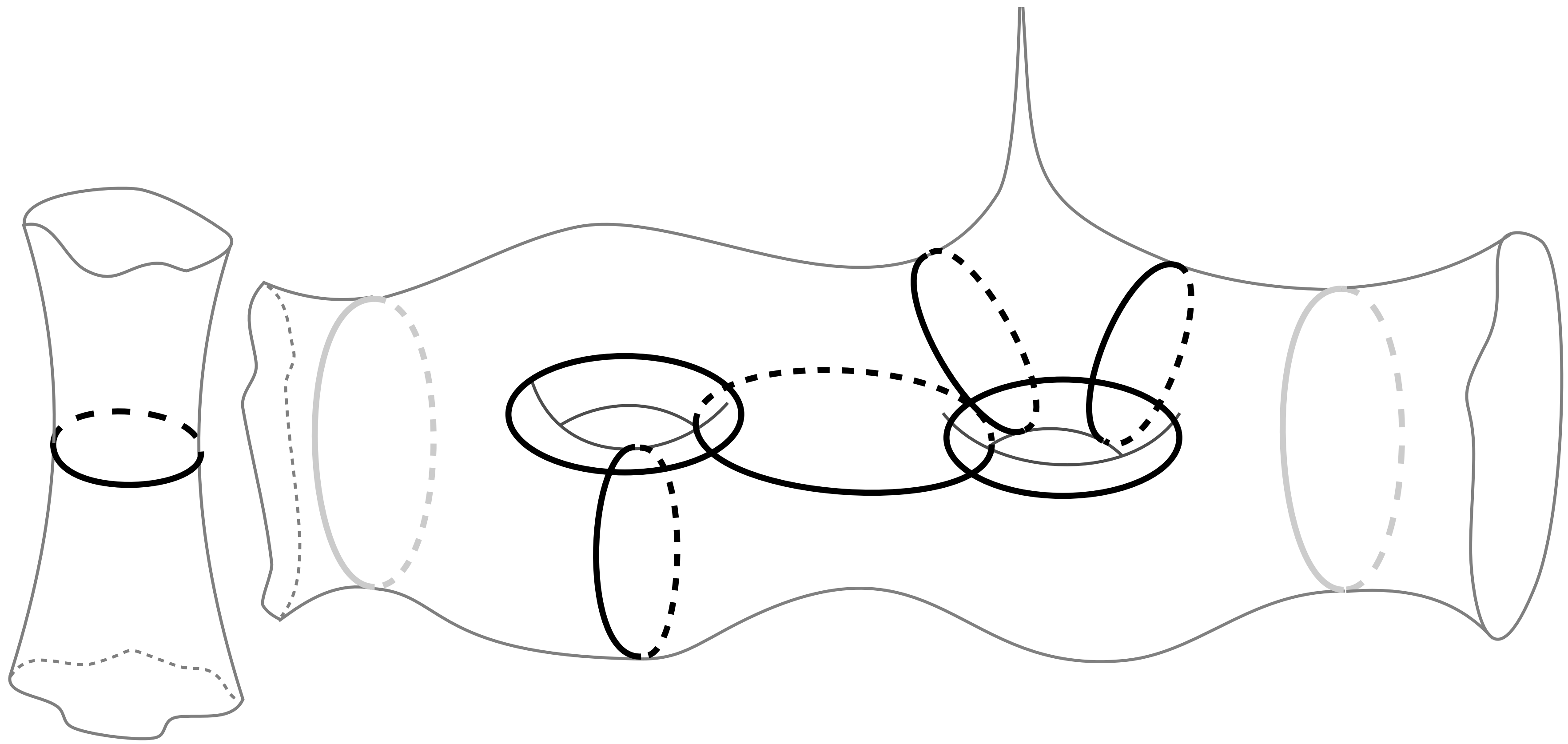
Observation (P.- Zappa using Saldanha-Tomei-Casarin-Romualdo)

The diameter of the flip graph of an n by n square (n even) is

$$\frac{n^3 - n}{6}$$

Secret goal in life:

Understand all 2-dimensional manifolds

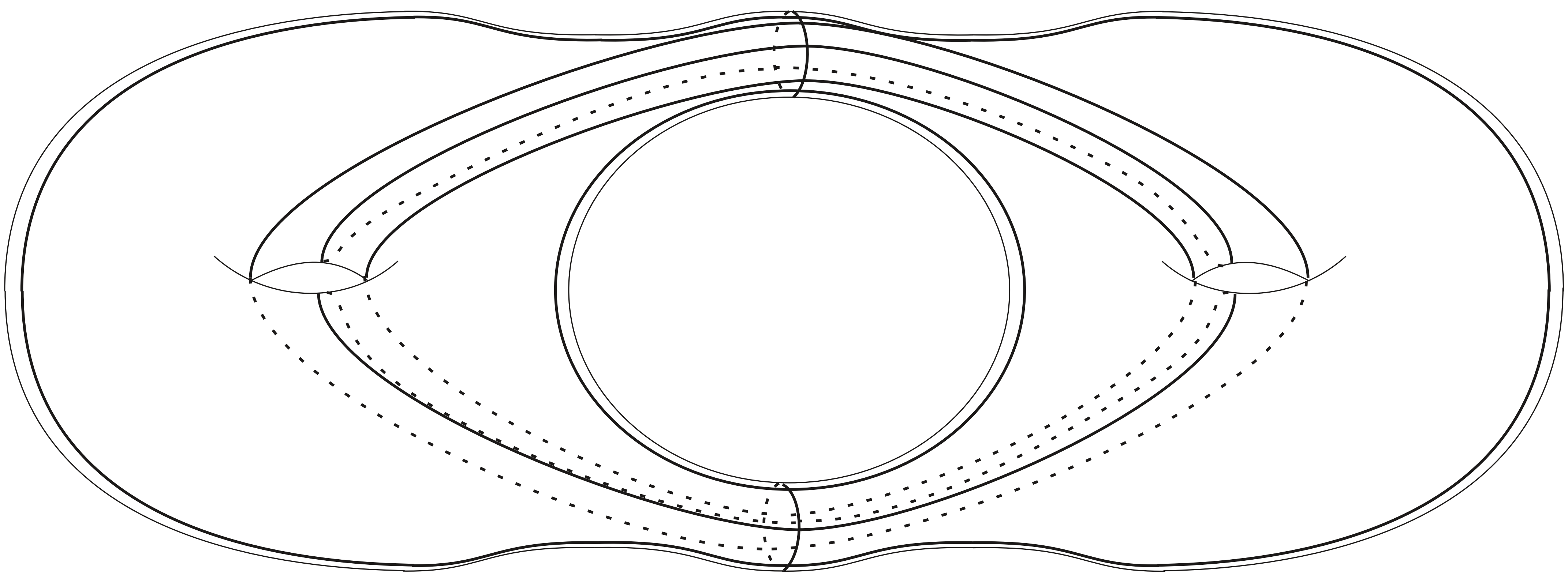


Moduli spaces:

Configuration spaces of conformal structures on surfaces up to biholomorphic equivalence

Smooth world:

Hyperbolic structures on surfaces

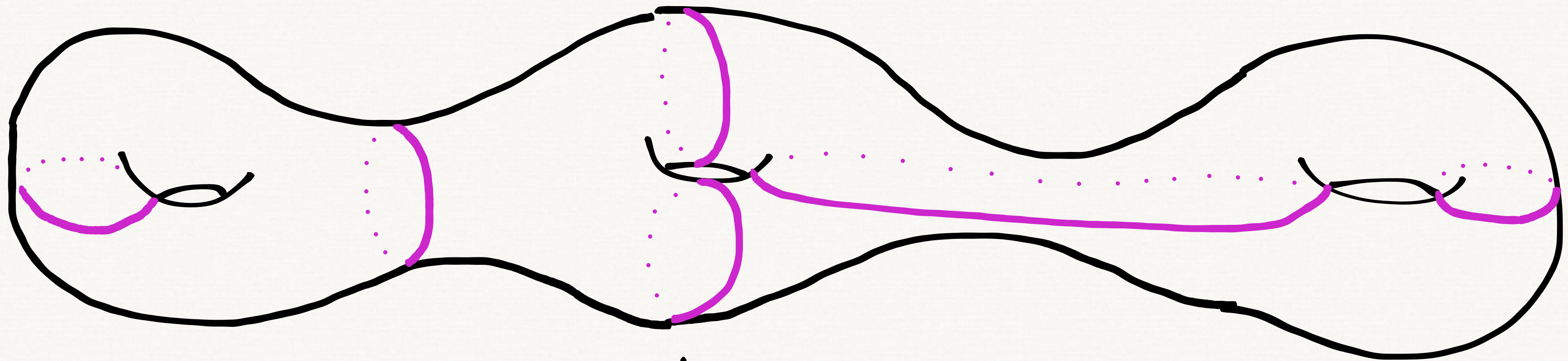


Moduli space:

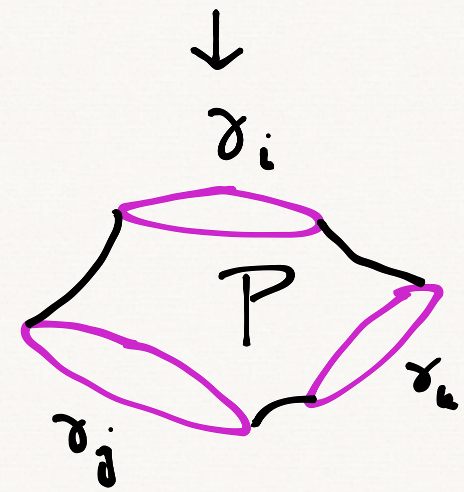
M_g is the space of hyperbolic surfaces of genus g up to isometry

Parameters for hyperbolic surfaces

X hyperbolic :

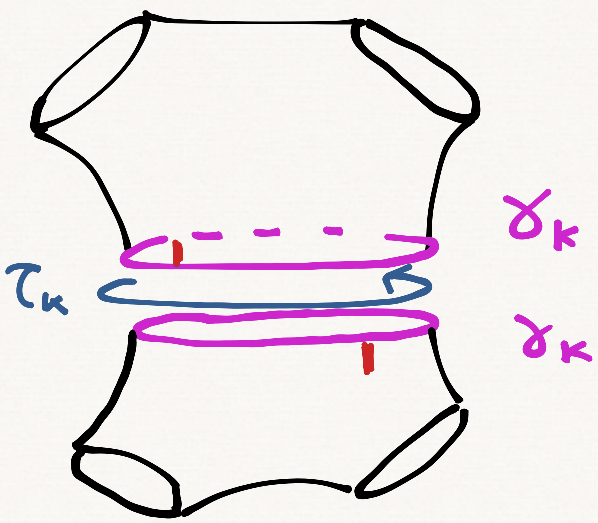


$\gamma_1, \dots, \gamma_{3g-3}$ geodesic pants decomposition



$\ell(\gamma_i), \ell(\gamma_j), \ell(\gamma_k)$ determine P

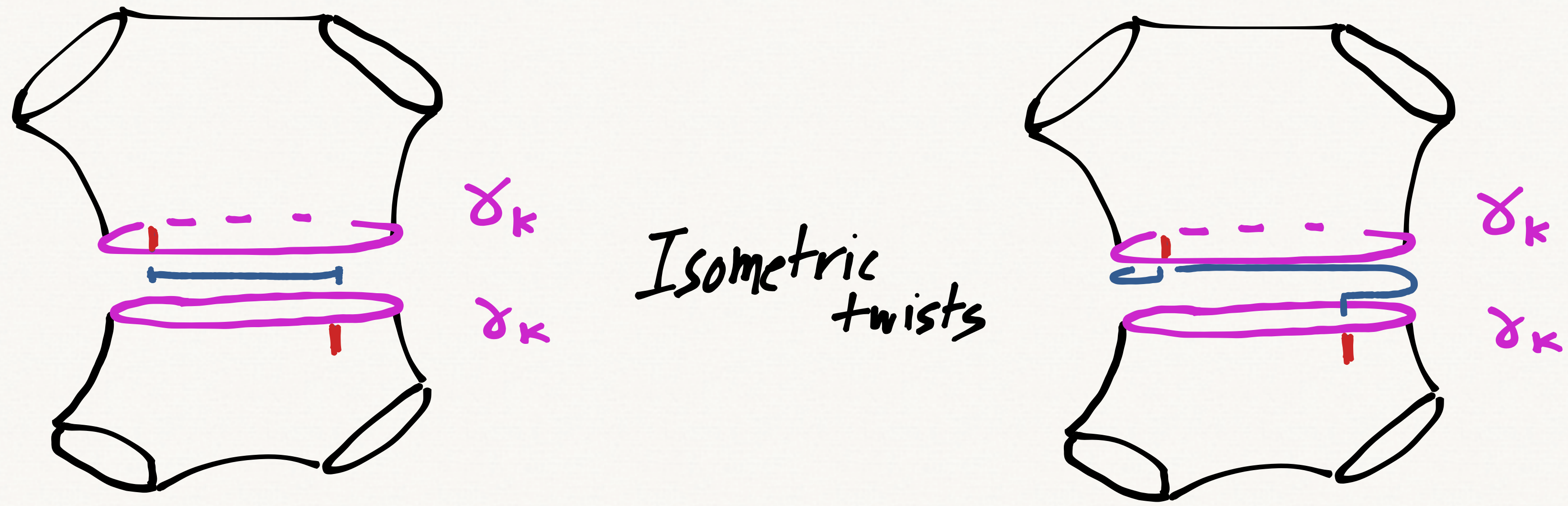
Twist parameter



$$\ell_1 := \ell(\gamma_1), \dots, \ell_{3g-3} := \ell(\gamma_{3g-3}) \in \mathbb{R}^{>0}$$

$$\tau_1, \dots, \tau_{3g-3} \in \mathbb{R}$$

Teichmüller space $\mathcal{T}_g$: $(\mathbb{R}^{>0})^{3g-3} \times \mathbb{R}^{3g-3}$



$$\mathcal{T}_g / \text{isometry}^+ = \mathcal{T}_g / MCG(\Sigma_g) = \mathcal{M}_g$$

Metric study of $\mathcal{M}_g$ via $MCG(\Sigma_g)$-invariant metrics on $\mathcal{T}_g$

Weil-Petersson metric:

Kähler metric on T_g defined on the co-tangent space *Q(X)*:

Let *X* be a point of Teichmüller space with line element ρ and *Q(X)* be the space of holomorphic quadratic differentials (i.e., locally of the form $h(z)dz^2$).

For φ, ψ ∈ *Q(X)* we obtain a Hermitian inner product given by

$$< \varphi, \psi >:= \int_X \frac{\varphi\overline{\psi}}{\rho^2}$$

The Weil-Petersson metric is given by taking the real part.

Weil-Petersson metric:

Volume is very natural: given by the standard volume with Fenchel-Nielsen coordinates described before:

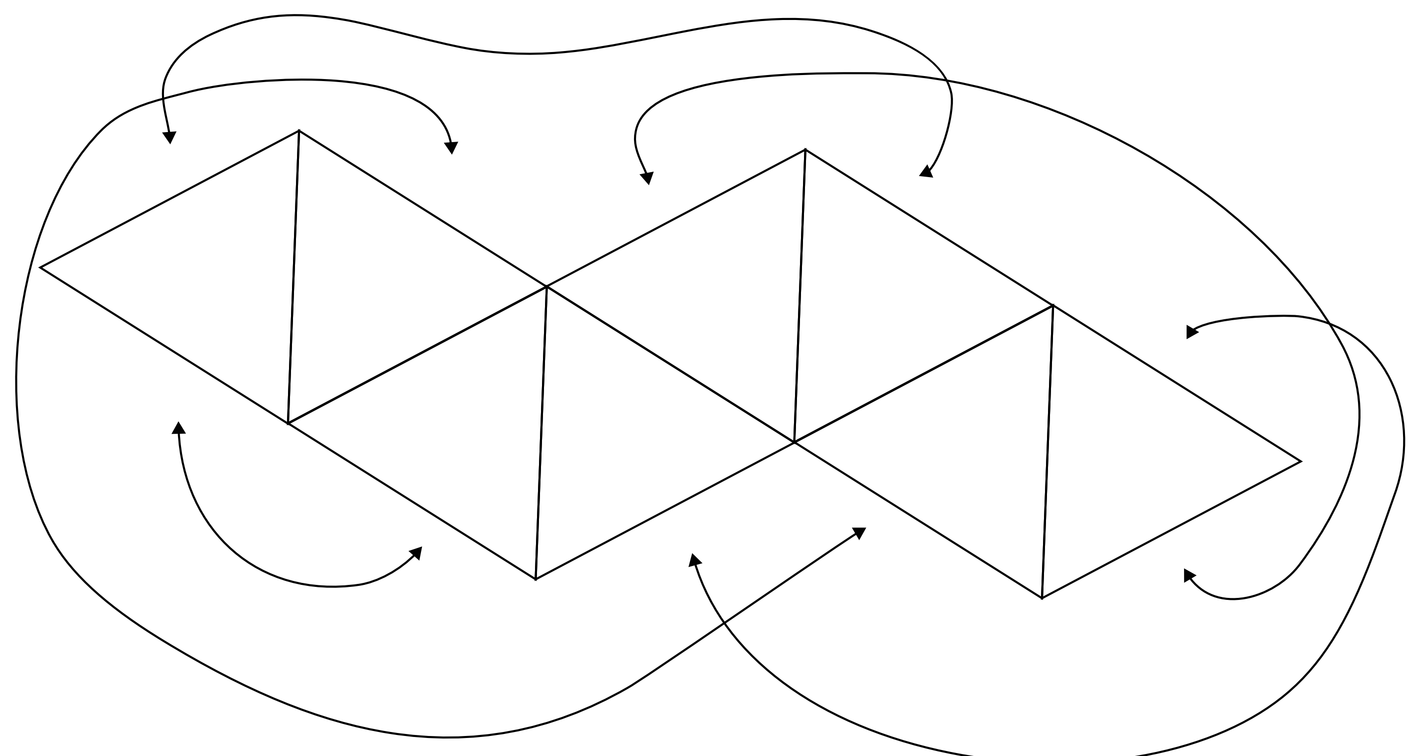
$$dV = dl_1 \wedge \ldots \wedge dl_{3g-3} \wedge d\tau_1 \wedge \ldots \wedge d\tau_{3g-3}$$

Smooth world:

Hyperbolic structures on surfaces

Combinatorial world:

Surfaces obtained by gluing triangles

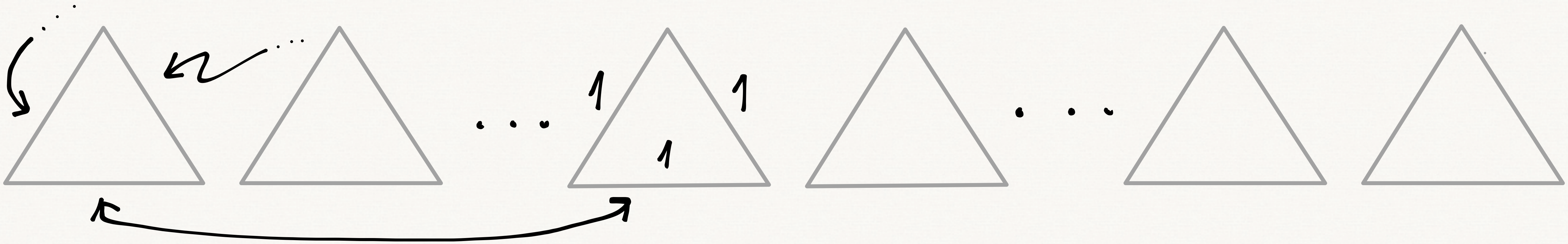


Moduli space:

M_g is the space of hyperbolic surfaces of genus g up to isometry

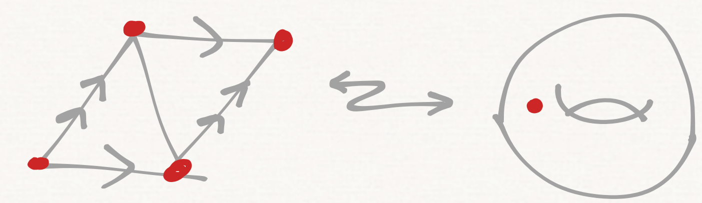
Combinatorial moduli space:

C_{2N} is the set of combinatorial surfaces with 2N triangles up to isomorphism



Collection of $2N$ randomly glued together Euclidean triangles of side length 1 such that:

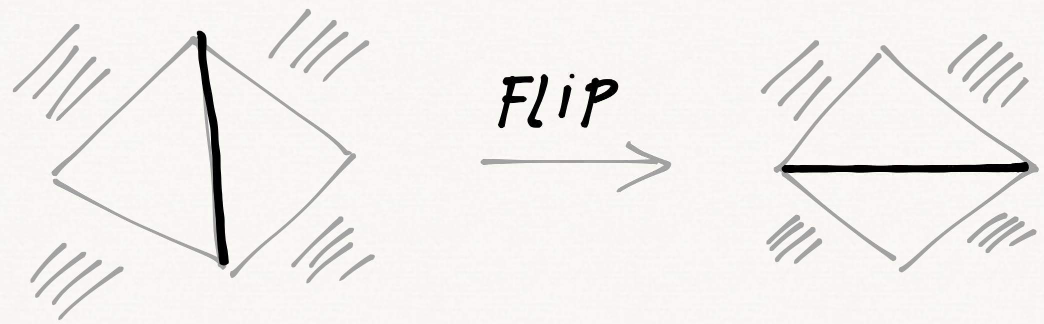
- result is orientable
- (slight simplification) there is only 1 vertex



$$\boxed{\text{Denote} \quad \mathcal{C}_{2N} = \{\text{this set}\} \big/ \text{simplicial automorphism}}$$

Volume $\longleftrightarrow$ Cardinality

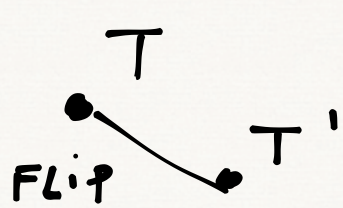
Distance?



For $T, T' \in \mathcal{T}_{2N}$:

$$\text{distance}(T, T') := \text{min \# of flips between } T \text{ and } T'$$

Flip graph



associated to $\mathcal{T}_{2N}$ is always connected

Size and shape of moduli space:

Theorem (Schumacher-Trapani)

$$\mathrm{Vol}(\mathcal{M}_g) \approx g^{2g}$$

where $\approx$ is up to exponential function in g.

Theorem (Cavendish-P.)

$$\mathrm{diam}(\mathcal{M}_g) \approx \sqrt{g}$$

where $\approx$ is up to logarithmic function in g.

Size and shape of combinatorial moduli space:

Theorem (Bollobás, Penner)

$$\mathrm{Card}(\mathcal{C}_{2N}) \approx g^{2g}$$

where $\approx$ is up to exponential function in g.

Theorem (Disarlo-P.)

$$\mathrm{diam}(\mathcal{C}_{2N}) \approx g \log(g)$$

where $\approx$ is up to universal multiplicative constants.

Random surfaces in $\mathcal{M}_g$:

Have "expander type"* properties (Mirzakhani)
small systole (Mirzakhani)
but large pants** (Guth-P.-Young)

Random surfaces in $\mathcal{C}_{2N}$:

Have "expander type" properties (Kolmogorov/Brooks-Makover)
small systole (Petri)
but large pants (Guth-P.-Young)

* *Expander type means* $\lambda_1>c>0$

** *Large pants means all pants decompositions of length at least* $g^{7/6-\varepsilon}$

Work in progress with Guth-Young:

The moduli space of genus g surfaces of area approximately A and bounded geometry* at scale 1 is connected.

** Bounded geometry at scale 1 means norm of curvature at most 1 and injectivity radius at least 1*

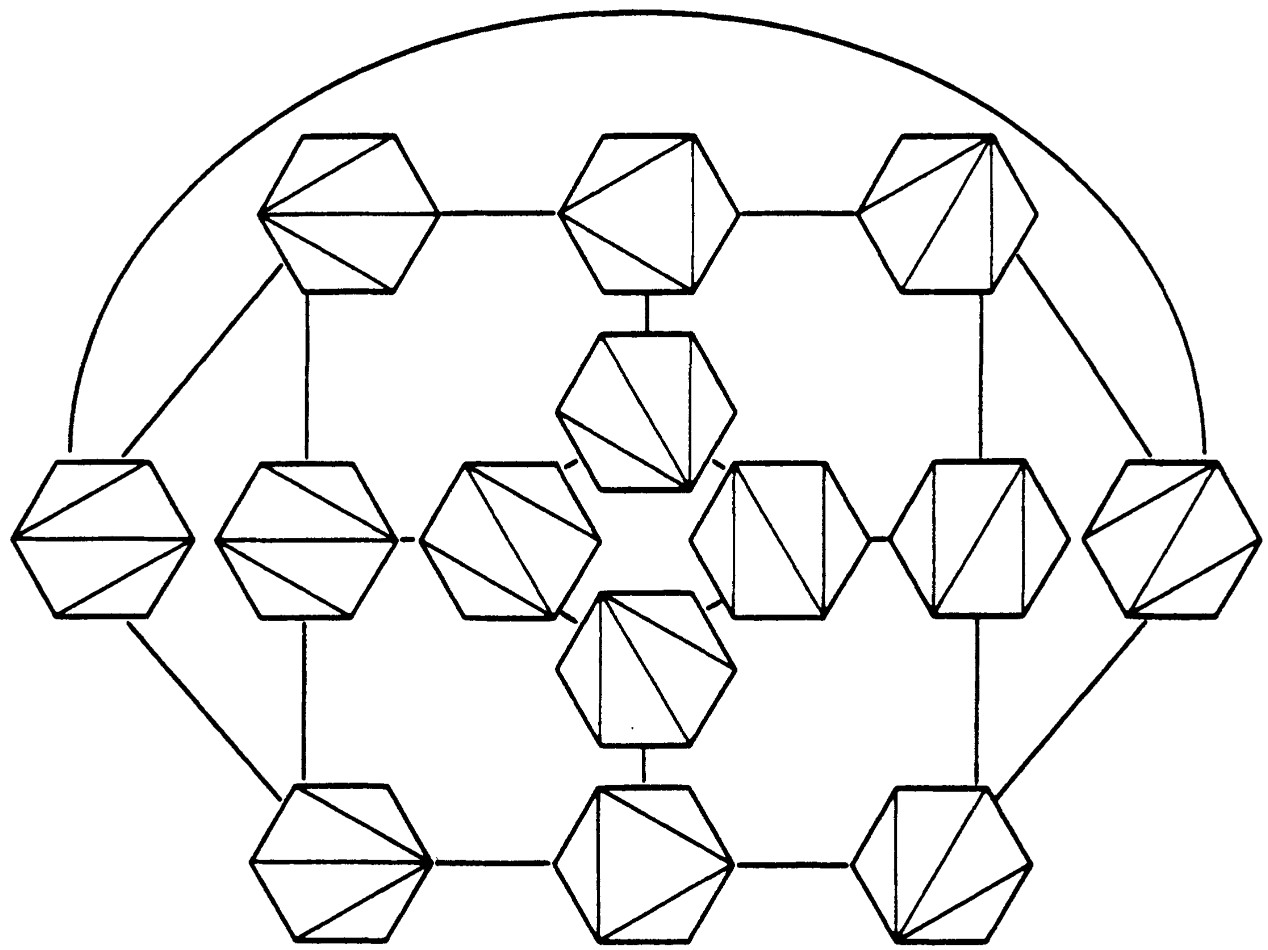


FIGURE 4. The rotation graph of a hexagon, $RG(6)$.

Flip graphs of polygons:

Sleator-Tarjan-Thurston (1988)

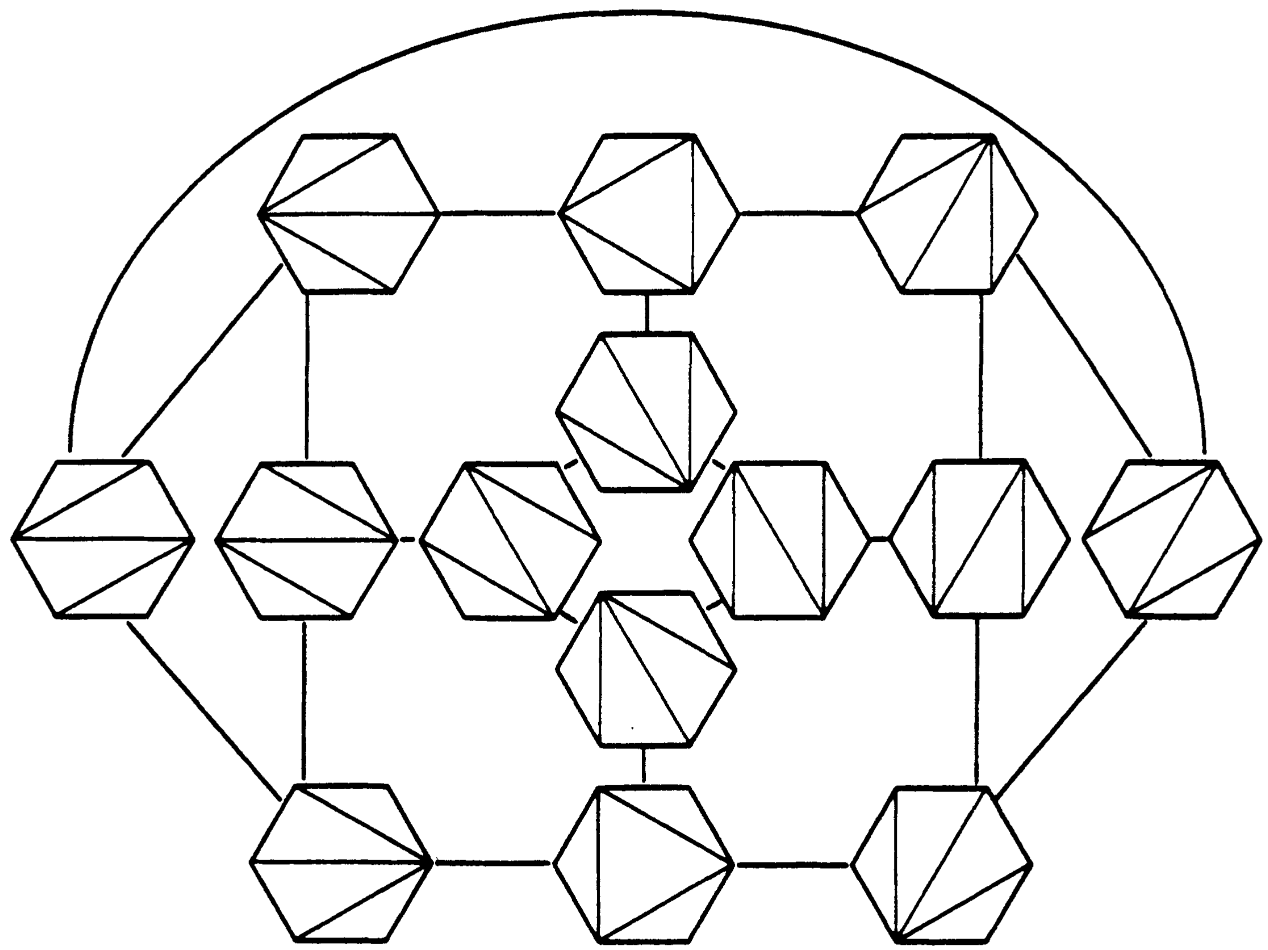


FIGURE 4. The rotation graph of a hexagon, $RG(6)$.

Theorem (Sleator-Tarjan-Thurston)
For sufficiently large n, the diameter of the flip graph of an n-gon is 2n-10.

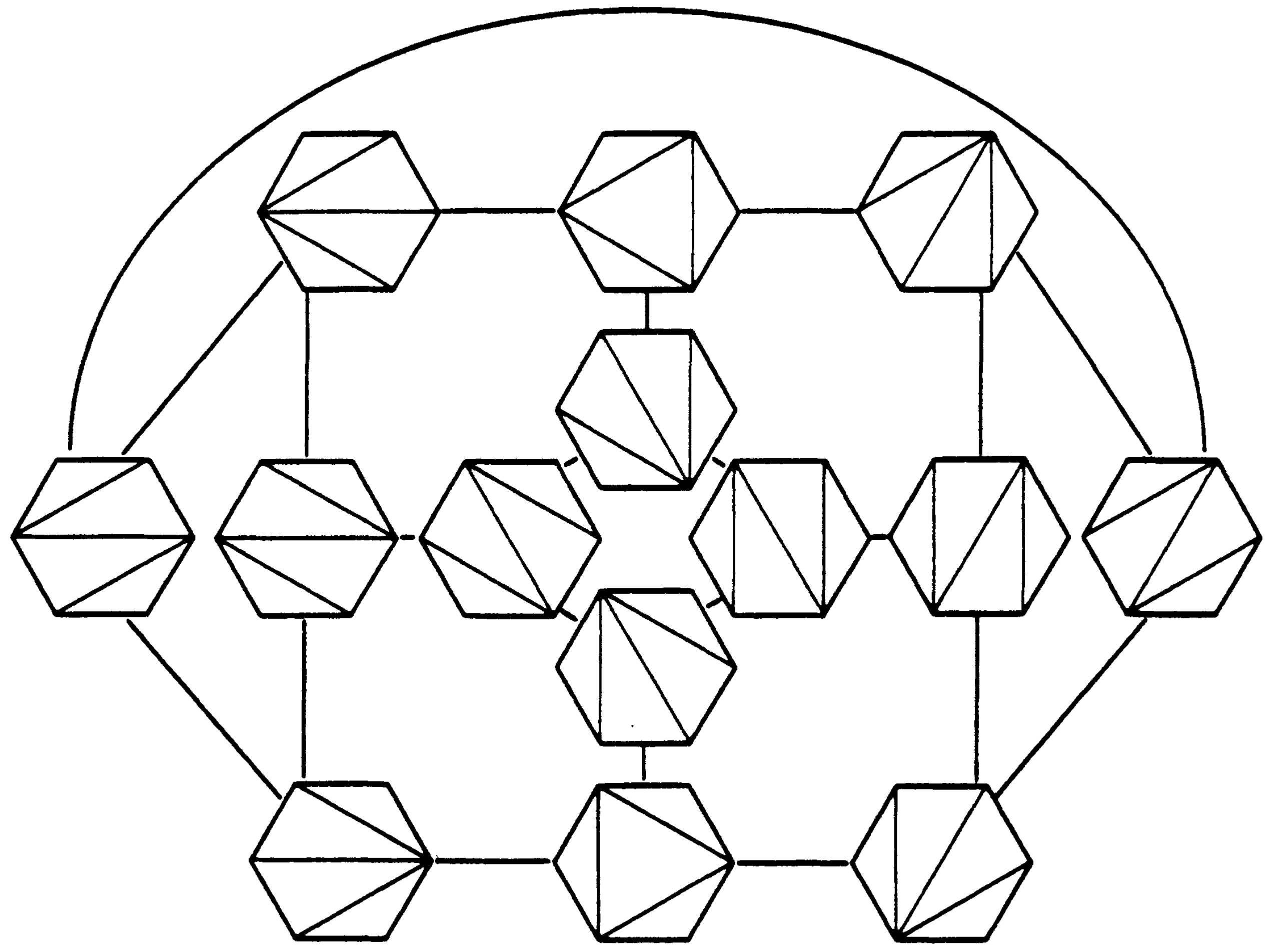
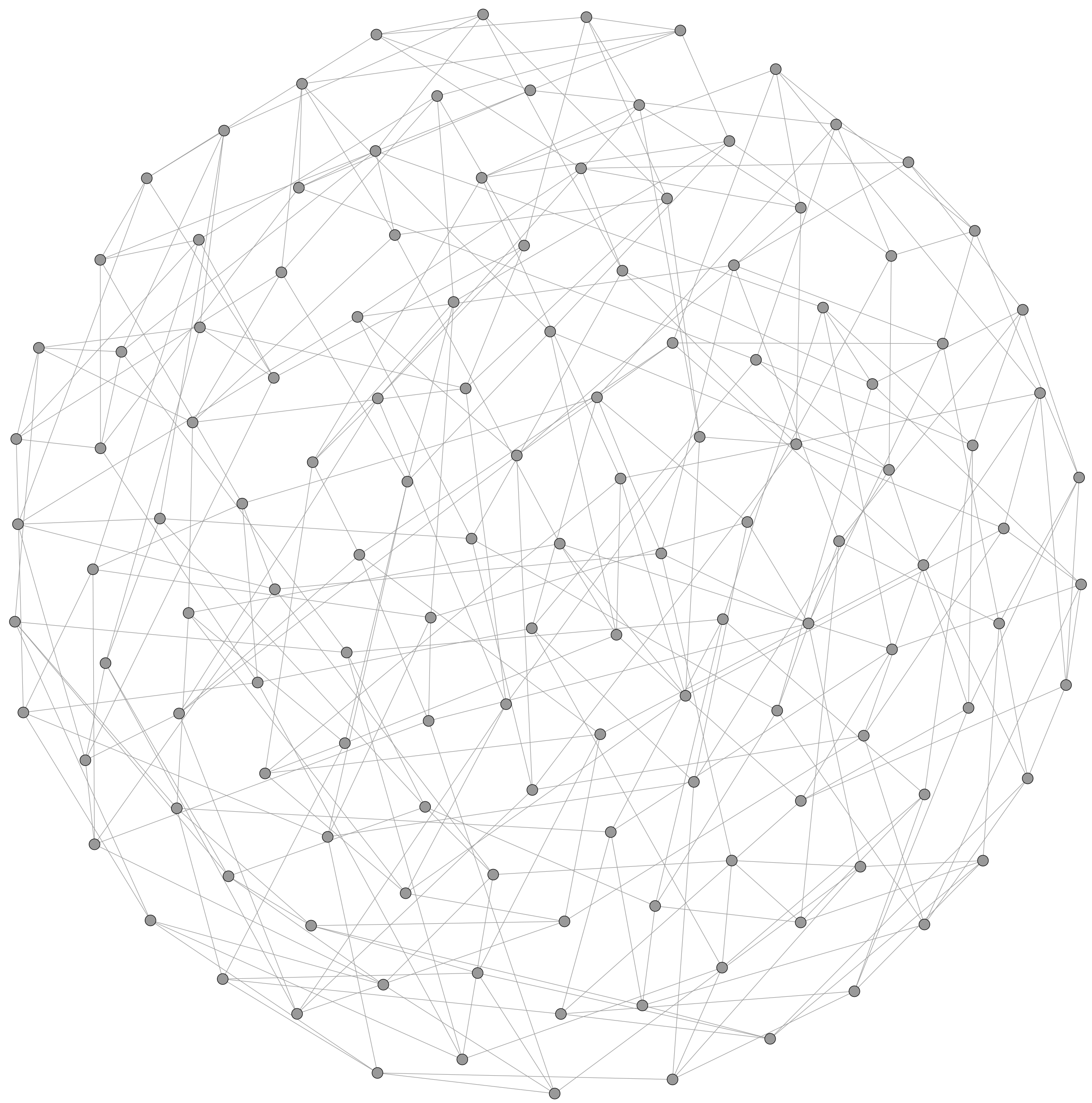


FIGURE 4. The rotation graph of a hexagon, $RG(6)$.

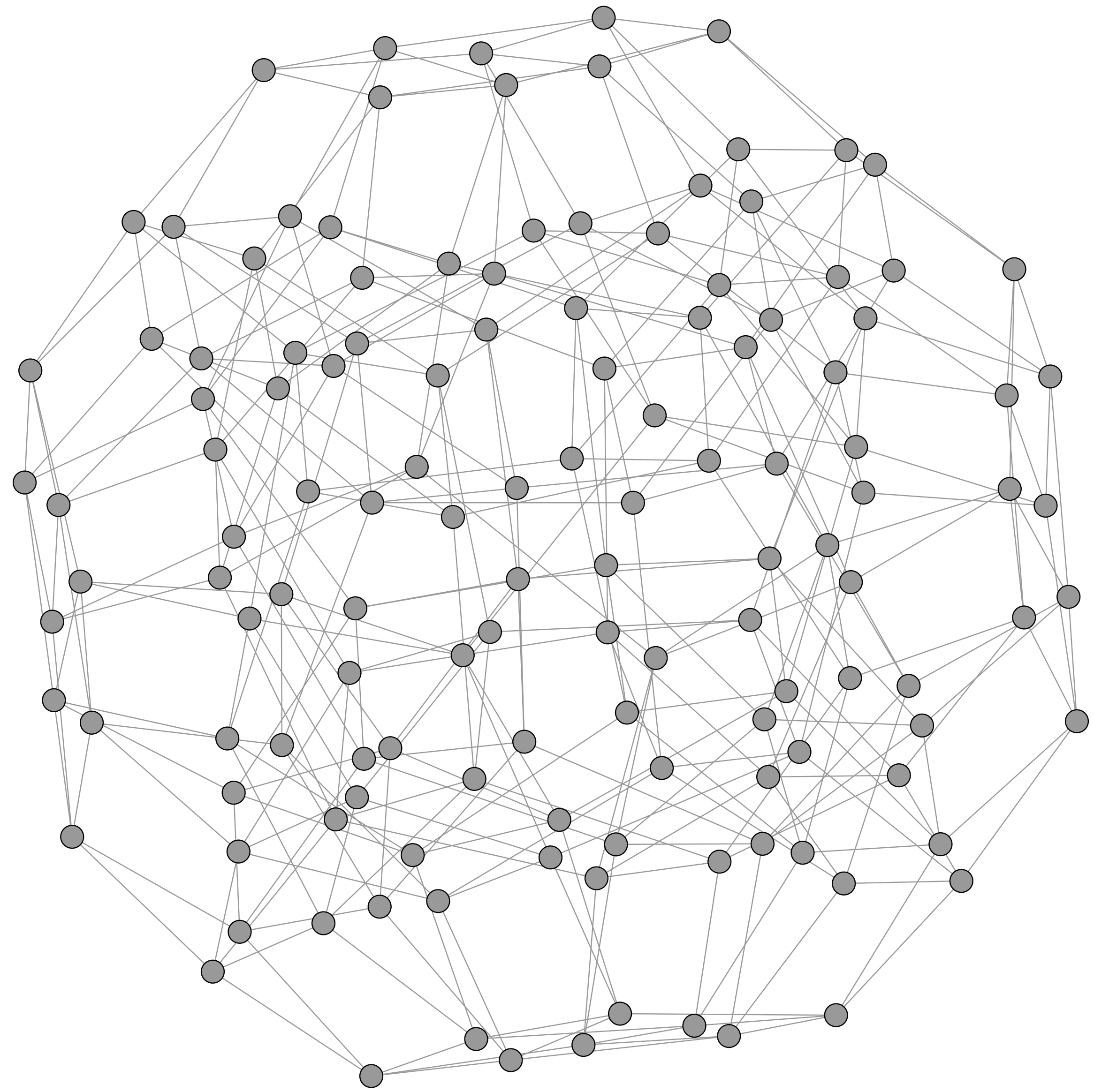
Theorem (Pournin)

For n>12 the diameter of the flip graph of an n-gon is 2n-10.



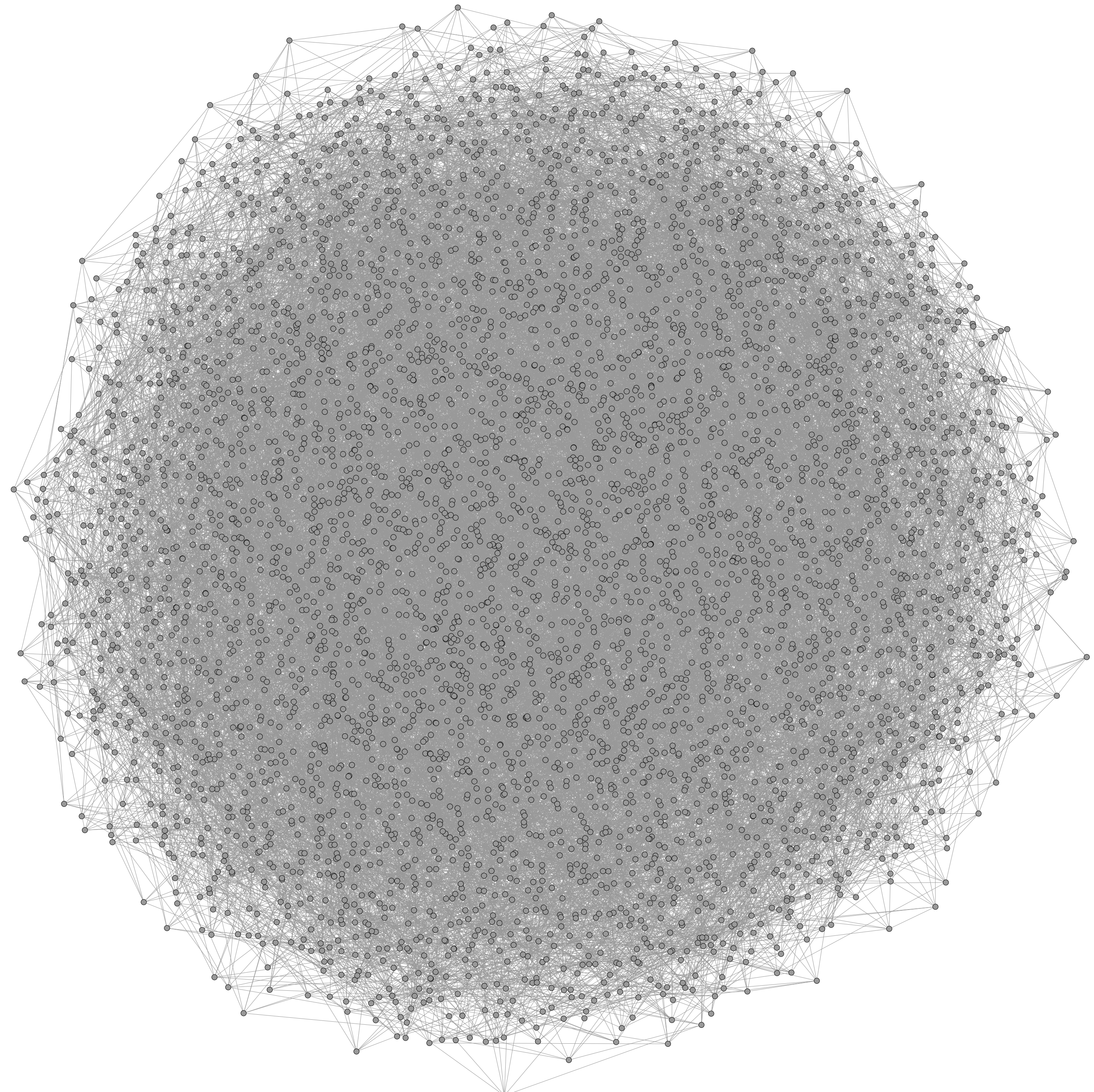
The graph for n=8 (Project with Bell and Pournin)

Graph computed using Mark Bell's program "Flipper", visualized using "Gephi"



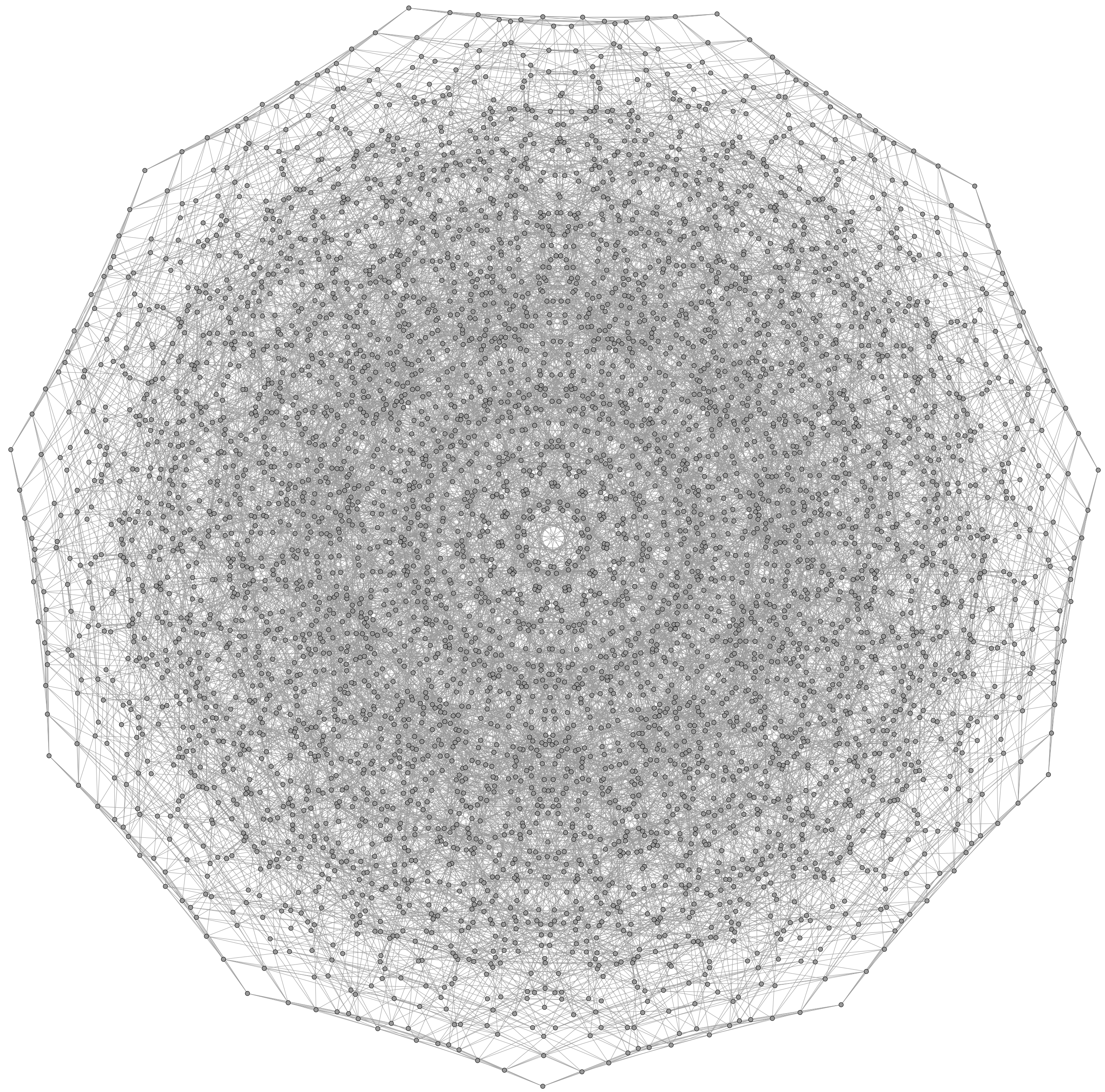
The graph for n=8

Re-arranged using an algorithm developed by Yifan Hu



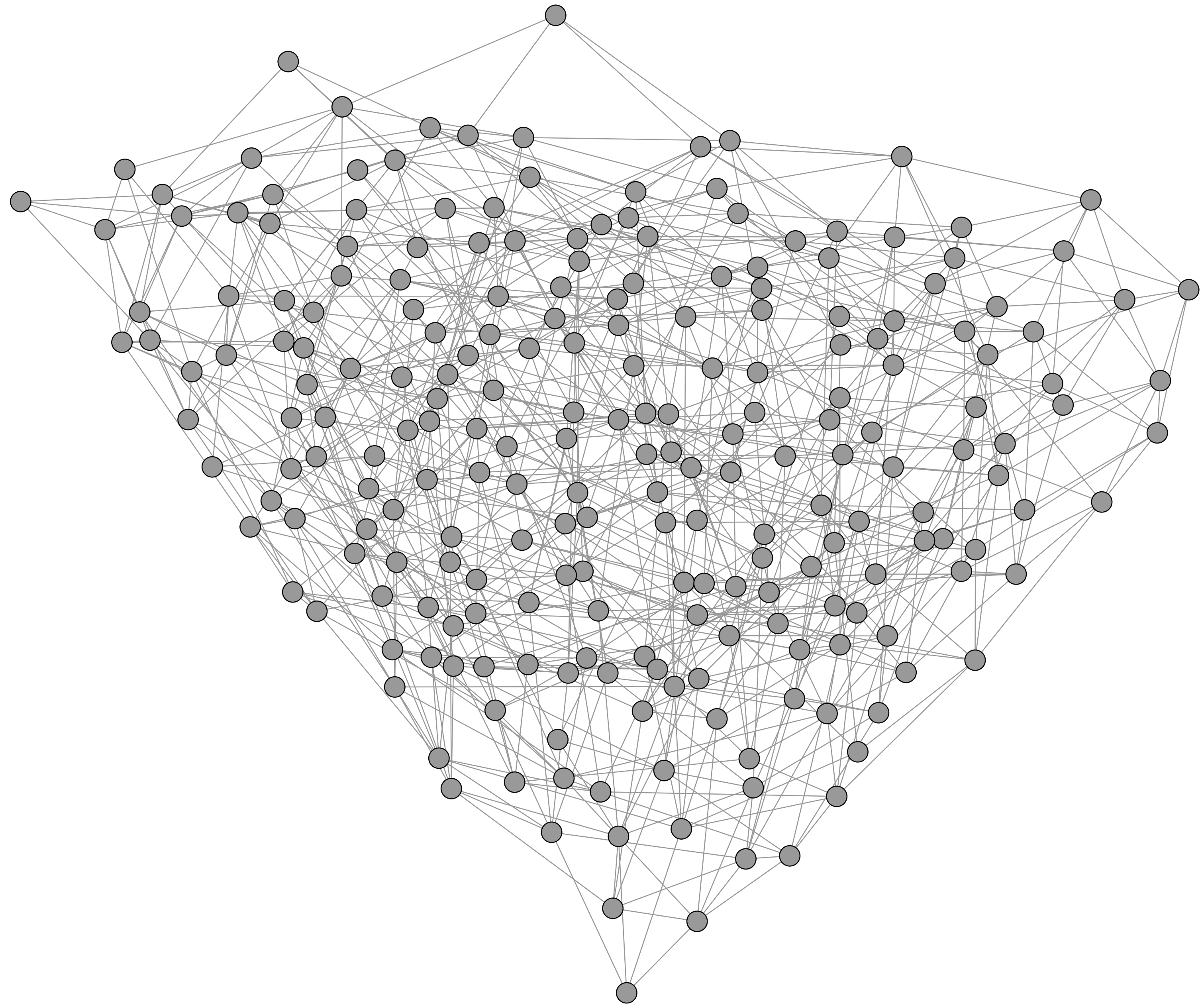
The graph for n=11

Graph computed using Mark Bell's program "Flipper", visualized using "Gephi"



The graph for n=11

Re-arranged using an algorithm developed by Yifan Hu



The modular graph for n=11