Ricci flow from spaces with isolated conical singularities

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What is Ricci flow?

Let M be a compact manifold, $\partial M = \emptyset$. A family $(g(t))_{t\in[0,T)}$ of Riemannian metrics on M is a Ricci flow (RF) if

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- Analogue of the heat diffusion equation, $\frac{\partial}{\partial t} u = \Delta u$, for Riemannian metrics. In fact, in harmonic coordinates, i.e. $\Delta_g x^i = 0$,

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$$-2\,\mathrm{Ric}(g)_{ij} = \Delta_g g_{ij} + Q(g, \partial g)_{ij}.$$
- Expectation: Ricci flow tries to find 'optimal' geometry on M and maximize amount of symmetry. Justified by examples.

Examples

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- But need to deal with singularities.

Singularities

Ricci flow is non-linear and existence of ‘optimal’ geometry depends on the topology of M. The flow typically develops singularities: smooth flow exists on a maximal time interval $[0, T)$, $T < \infty$ and

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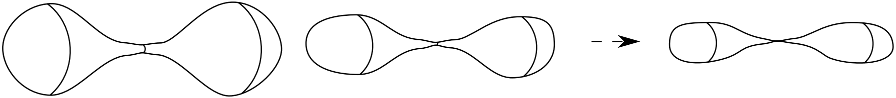
- Neck-pinch singularity on S^{n+1}, $n \geq 2$. (Angenent–Knopf):



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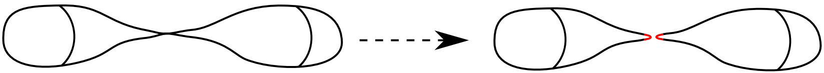

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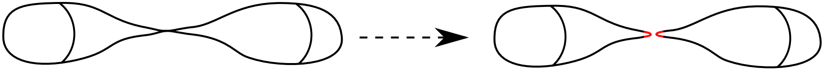

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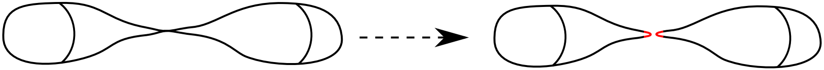

- In general, much more complex behaviour.
- Feldman–Ilmanen–Knopf: Example of non-compact smooth Ricci flow that converges to a cone, and then changes topology, becomes smooth and continues after the singularity.

Geometry of singularities

Theorem (Naber, Enders–Müller–Topping)

Let $(g(t))_{t\in[-T,0)}$ be a Type I Ricci flow and $p \in M$ be a singular point. Then for any $\lambda_k \searrow 0$ the dilated flows $(M, \lambda_k^{-1} g(\lambda_k t), p)_{t\in[-\lambda_k^{-1}T,0)}$ converge, as $k \to \infty$, to a non-flat flow $(N, h(t))_{t\in(-\infty,0)}$ which has special structure: gradient shrinking Ricci soliton.

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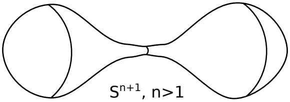

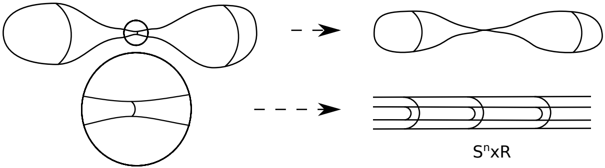

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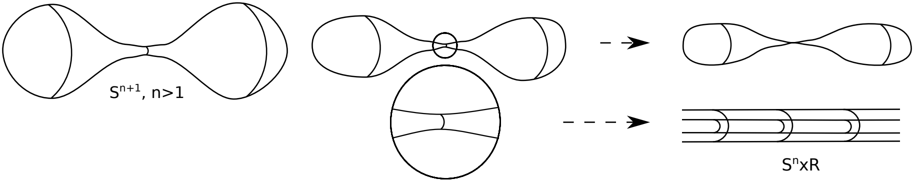

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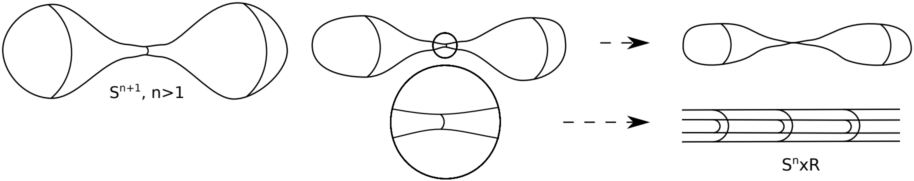

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- Think of a GSRS as a fixed point of the flow, modulo diffeomorphisms and scalings.

Conical structure of singularities

- If a singular Ricci flow $(M, g(t))_{t\in[0,T)}$ converges to a singular space (X, d_X) as $t \to T$, we would like to be able to restart the flow, with (X, d_X) as initial data.

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- The structure of X is far from being understood.
- Evidence that its singularities have conical structure.

Theorem (Munteanu–Wang)

Let $(M^n, \mathbf{g}, f)$ be a non-compact GSRS with $\mathrm{Ric} \to 0$ at infinity. Then it is asymptotic to the cone $(C(X), g_c)$ over a closed $(n-1)$-dimensional Riemannian manifold (X, g_X), namely $C(X) = (0, +\infty) \times X$ and $g_c = dr^2 + r^2 g_X$.

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- Asymptotically conical *expanding* Ricci solitons: special solutions of Ricci flow coming out of cones.

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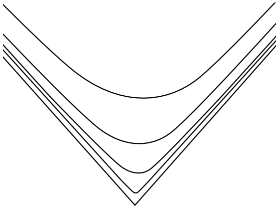

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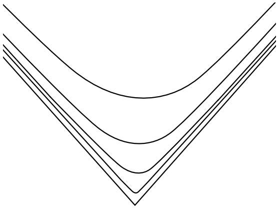

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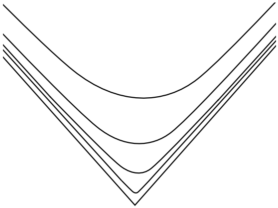

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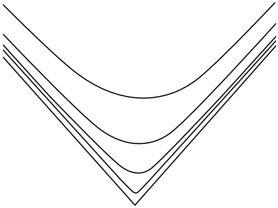
- Deruelle: $\mathrm{Rm} \geq 0$, asymptotic to cones $C(S^{n-1})$ with $\mathrm{Rm}(g_{S^{n-1}}) \geq 1$.



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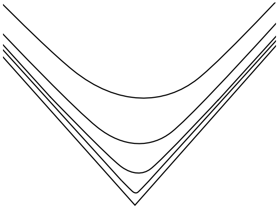
- H-D. Cao, Feldman–Ilmanen–Knopf: asymptotic to $\mathbb{C}^n/\mathbb{Z}_k$.



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Let (M, g_0) *be a Riemannian manifold, smooth away from* $q_0 \in M$*, and a conical singularity modeled on* $(C(S^{n-1}), g_c = dr^2 + r^2 g_{S^{n-1}})$ *at* q_0*.*

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4. *For any $\lambda_k \searrow 0$ and $q \notin \mathrm{Im}\Psi$, the sequence $(M, \lambda_k^{-1} g(\lambda_k t), q)_{t \in (0, \lambda_k^{-1} T]}$ converges smoothly to the flow generated by the unique expander asymptotic to the cone $(C(S^{n-1}), g_c)$.*

Idea of proof

- By a result of Deruelle, given $C(S^{n-1})$ with $\mathrm{Rm}(g_c) > 0$, there is a unique EGRS $(N, \mathbf{g}, f)$ asymptotic to $C(S^{n-1})$ at infinity, $N \equiv \mathbb{R}^n$, and $\mathrm{Rm}(\mathbf{g}) > 0$.

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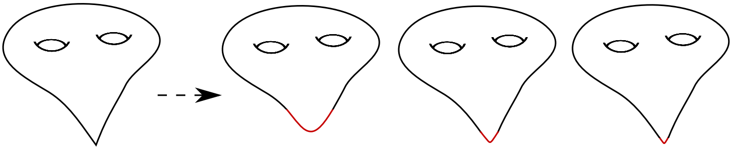

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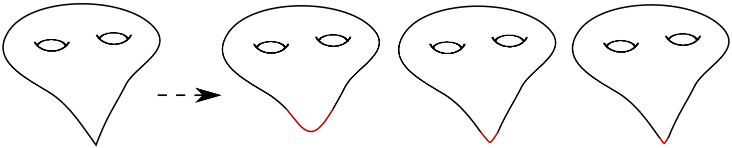

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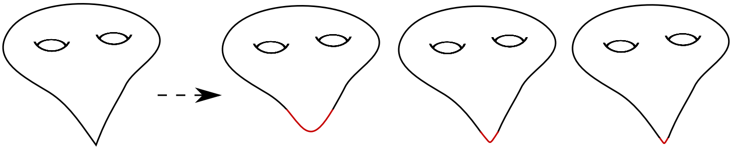

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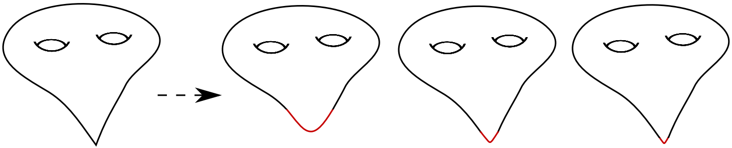
- Since $|\mathrm{Rm}(g_{0,s_i})| \sim s_i^{-1}$ standard theory only gives $T_i \sim s_i$.

Idea of proof

- By a result of Deruelle, given $C(S^{n-1})$ with $\mathrm{Rm}(g_c) > 0$, there is a unique EGRS $(N, \mathbf{g}, f)$ asymptotic to $C(S^{n-1})$ at infinity, $N \equiv \mathbb{R}^n$, and $\mathrm{Rm}(\mathbf{g}) > 0$.
- Desingularize (M, g_0) by gluing large pieces of $(N, \mathbf{g})$ at small scales $s_i \searrow 0$ to obtain (M, g_{0,s_i}).



- Consider the associated sequence of Ricci flows $(M, g_{s_i}(t))_{t\in(0,T_i]}$ with $g_{s_i}(0) = g_{0,s_i}$ and obtain uniform estimates.
- Since $|\mathrm{Rm}(g_{0,s_i})| \sim s_i^{-1}$ standard theory only gives $T_i \sim s_i$.
- Pass to a limit flow $(g(t))_{t\in(0,T]}$.

Perelman's pseudolocality theorem

Theorem

There exists $\varepsilon(n) > 0$ *such that if* $(M^n, g(t))_{t\in[0,(\varepsilon r)^2]}$ *is a complete Ricci flow and*

$$\begin{aligned} \mathrm{vol}_{g(0)}(B_{g(0)}(x,r)) &\geq (1-\varepsilon)\omega_n r^n, \\ |\,\mathrm{Rm}(g(0))|_{g(0)} &\leq r^{-2}, \quad \text{in } B_{g(0)}(x,r), \end{aligned}$$

then $|\,\mathrm{Rm}(g(t))|_{g(t)} \leq (\varepsilon r)^{-2}$ *in* $B_{g(0)}(x, \varepsilon r)$ *and* $t \in [0, (\varepsilon r)^2]$.

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- Obtain control in regions with 'almost Euclidean' volume.

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- For some time, extreme high curvature outside $B_{g(0)}(x,r)$ does not influence the flow in $B_{g(0)}(x,\varepsilon r)$ significantly.

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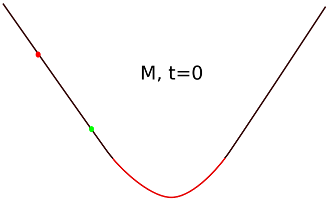
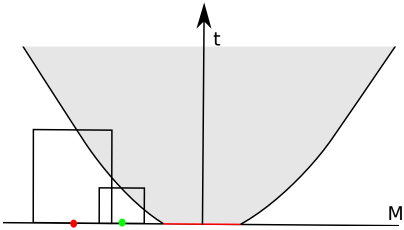
- Obtain control in regions with 'almost Euclidean' volume.
- For some time, extreme high curvature outside $B_{g(0)}(x,r)$ does not influence the flow in $B_{g(0)}(x, \varepsilon r)$ significantly.
- That is very different from the behaviour of solutions of the heat equation!

Estimating the flow away from the singular point

- Apply the pseudolocality theorem to the approximators $(M, g_{s_i}(t))_{t\in[0,T_i]}$ away from the red region, where (M, g_{0,s_i}) is close to the cone $(C(S^{n-1}), g_c)$.





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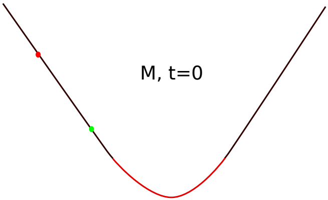

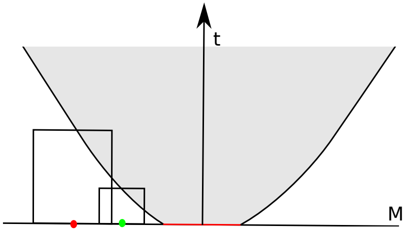

- By the Ricci flow equation, $g_{s_i}(t)$ remains close to $g(0)$ outside the grey, 'expanding region' (uniformly as $s_i \to 0$).

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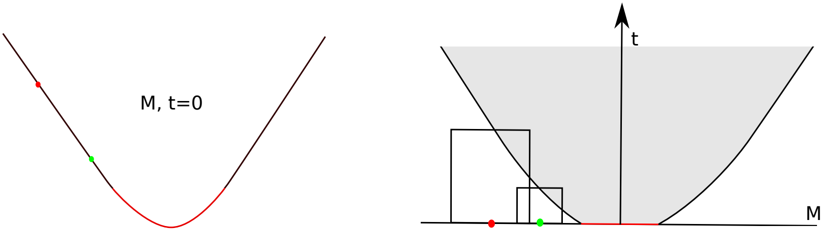

- By the Ricci flow equation, $g_{s_i}(t)$ remains close to $g(0)$ outside the grey, 'expanding region' (uniformly as $s_i \to 0$).
- In the expanding region, we expect that $g_{s_i}(t)$ remains close to the evolution of the expander $g_e(t + s_i)$, for $t \geq 0$.

Estimating the flow in the expanding region

Localize a stability result of Deruelle–Lamm for the closely related Ricci–DeTurck equation on expander $(N, g_e(t+1))$ with $\mathrm{Rm} > 0$

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t)) + \mathcal{L}_{\mathcal{W}(g(t), g_e(t+1))} g(t).$$

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- Use the control obtained by the pseudolocality theorem to obtain control in the annular region.

Estimating the flow in the expanding region

Localize a stability result of Deruelle–Lamm for the closely related Ricci–DeTurck equation on expander $(N, g_e(t+1))$ with $\mathrm{Rm} > 0$

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- Use the control obtained by the pseudolocality theorem to obtain control in the annular region.
- Technical difficulty: the Ricci flow is related to the Ricci–DeTurck flow via a family of diffeomorphisms which we need to control.

THANK YOU!