Weak Lefschetz Property and stellar subdivisions of simplicial complexes

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Motivation and Background

MAIN MOTIVATION The g-Conjecture, which is a conjecture about the combinatorics of a (finite) simplicial complex D triangulating a sphere S^n.

In more detail, assume D is a finite simplicial complex. We define $f_i(D)$ to be the number of i-faces of D. (A 0-face is also called a vertex, a 1-face is also called an edge.)

Assume the dimension of D is $d-1$. We define the f-vector $f(D)$ of D by

$$f(D) = (f_0(D), f_1(D), \ldots, f_{d-1}(D))$$

EXAMPLE: If D' is the solid triangle, we have

$$f(D') = (3, 3, 1),$$

since it has 3 vertices, 3 edges and one 2-face.

QUESTION Assume $n \geq 1$. Denote by $\mathcal{A}_n$ the set of all simplicial complexes D triangulating S^n. Describe the set

$$f(\mathcal{A}_n) = \{f(D) : D \in \mathcal{A}_n\}.$$

EXAMPLE: For $n = 1$ we only have m-gons with $m \geq 3$, hence

$$f(\mathcal{A}_1) = \{(m, m) : m \geq 3\}.$$

REMARK: For $D \in \mathcal{A}_n$ a linear relation is given by the topological Euler characteristic

$$\sum_{i=0}^{n} (-1)^i f_i(D) = \chi(S^n) = 1 + (-1)^n$$

Another relation is $f_0(D) \geq n + 2$.

g-CONDITIONS: A certain set of equalities and inequalities between the f_i, obtained in a different context by Macaulay (1920s) when studying the growth of Hilbert functions of graded rings.

ORIGINAL g-CONJECTURE (McMullen, 1971) A finite sequence

$$f = (f_0, \ldots, f_{n-1}) \in \mathbb{N}^n$$

is the f-vector of the boundary complex of a convex simplicial polytope P of dimension n if and only f satisfies the g-conditions.

REMARK: A polytope is called simplicial if all proper faces are simplices. For example the cube and the dodecahedron are not simplicial polytopes. The tetrahedron, octahedron and icosahedron are simplicial.

The original g-Conjecture is true:

Billera and Lee (1981) showed (explicit construction) that if a finite sequence f satisfies the g-conditions, then there exists a convex simplicial polytope P such that the f-vector of the boundary complex of P is equal to f.

Stanley (1981) using the Hard Lefschetz theorem for the singular cohomology of a projective toric variety associated to P showed the other direction: The f-vector of the boundary complex of a convex simplicial polytope P satisfies the g-conditions.

REMARK. For high n there are many elements of $\mathcal{A}_n$ that can not be obtained as boundary complex of a convex simplicial polytope.

(CURRENT) g-CONJECTURE. Assume $D \in \mathcal{A}_n$. Then the f-vector $f(D)$ of D satisfies the g-conditions.

The conjecture is open for high n.

Stanley-Reisner Ring

Assume k is an infinite field and D is a simplicial complex with vertex set $\{1, \ldots, m\}$ and dimension $d-1$. The Stanley–Reisner ideal $I_D \subset k[x_1, \ldots, x_m]$ of D is the ideal

$$I_D = (x_{i_1} x_{i_2} \cdots x_{i_p} : i_1 < i_2 \cdots < i_p \text{ and } \{i_1, i_2, \ldots, i_p\} \notin D).$$

It is a square-free monomial ideal.

The Stanley–Reisner ring of D is

$$k[D] = k[x_1, \ldots, x_m]/I_D.$$

We put $\deg x_i = 1$ for all i.

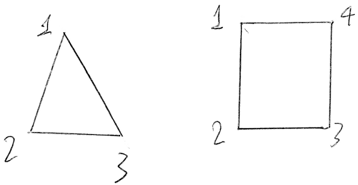
EXAMPLES

- If D is the 3-gon then

$$k[D] = k[x_1, x_2, x_3]/(x_1x_2x_3)$$

- Assume D is the 4-gon (i.e., boundary of the square) with diagonals $\{1, 3\}$ and $\{2, 4\}$. We have

$$k[D] = k[x_1, \ldots, x_4]/(x_1x_3, x_2x_4).$$



REMARKS
We have $\dim k[D] = \dim D + 1 = d$.

From the Hilbert Series of $k[D]$ we can recover the f-vector of D: Set

$$H_D(t) = \sum_{t=0}^{\infty} \dim_k(k[D]_i)t^i \in k[[t]]$$

i.e., considered as a formal power series. Then $(1-t)^d H_D(t)$ is a polynomial in t, and the information of its coefficients is equivalent to the information $f(D)$.

Moreover, by work of Hochster, Stanley and Reisner, good topological properties of the topological space $|D|$ are related to good algebraic properties of the ring $k[D]$. For example, if $D \in \mathcal{A}_n$ then $k[D]$ is a Gorenstein ring.

Weak Lefschetz Property

DEFINITION Assume $\dim k[D] = d$. We say that $k[D]$ has the Weak Lefschetz Property (WLP) if for $d+1$ (Zariski) general linear elements $f_1, \dots, f_{d+1}$ and for all i we have that

$$\dim_k(B_i) = \max\{\dim_k A_i - dim_k A_{i-1}, 0\}$$

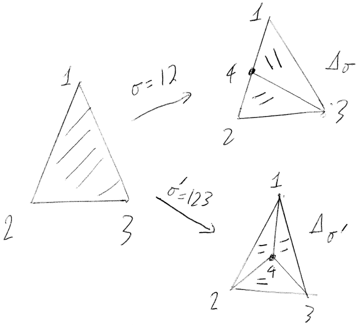
where $A = k[D]/(f_1, \dots, f_d)$ is a general Artinian reduction of $k[D]$ and $B = A/(f_{d+1})$.

REMARK Assume $D \in \mathcal{A}_n$. It is well-known that $k[D]$ WLP implies that the f-vector $f(D)$ of D satisfies the g-conditions.

QUESTION Assume $D \in \mathcal{A}_n$. Is it true that $k[D]$ WLP?

Stellar Subdivisions

Assume D is a simplicial complex and σ is a face of D. Then there is a new simplicial complex D_σ called the stellar subdivision of D with respect to σ.



EXAMPLE: If D is the m-gon and σ is a 1-face of D, then D_σ is the polygon with $m+1$ vertices.

MAIN THEOREM (Boehm - P.) Assume $D \in \mathcal{A}_n$ and σ is a face of D with $2 \dim \sigma > \dim D + 1$. Then $k[D]$ is WLP if and only if $k[D_\sigma]$ is WLP.

QUESTION Is the Main Theorem true without any restriction on $\dim \sigma$? If so, then it follows that: If D is a PL-sphere then the f-vector $f(D)$ of D satisfies the g-conditions.

Kustin–Miller unprojection

Kustin–Miller unprojection (Kustin and Miller (1985), Reid and P. (2000)) is a method for constructing and analyzing complicated Gorenstein rings in terms of simpler ones. In general, unprojection theory is the algebraic counterpart of certain constructions in birational geometry.

Assume R is a positively graded Gorenstein ring and $I \subset R$ is a homogeneous ideal such that $\dim R/I = \dim R - 1$, R/I is Gorenstein and I is not principle. Then standard facts about Gorenstein rings give that there exists a short exact sequence

$$0 \to R \to Hom_R(I, R) \to R/I \to 0$$

of R-modules. Hence, there exists an R-module homomorphism $\phi : I \to R$ such that ϕ together with the inclusion i generate $Hom_R(I, R)$ as R-module.

DEFINITION The Kustin–Miller unprojection ring S of the pair $I \subset R$ is the ring

$$S = R[T]/(Tu - \phi(u) : u \in I)$$

where T is a new variable. In a certain sense S is the ring corresponding to the "graph" of $\phi : I \to R$.

EXAMPLE: Assume k field, A, B are reasonably general homogenous polynomials in the polynomial ring $Q = k[x, y, z_1, z_2, \dots, z_p]$,

$$R = Q/(Ax - By), \qquad I = (x, y) \subset R$$

Then I is a codimension 1 ideal of R that is not principle. Moreover R and R/I are complete intersections, hence Gorenstein. We can choose the $\phi : I \to R$ such that

$$\phi(x) = B, \quad \phi(y) = A$$

(Since x, y generate the ideal I, this specifies ϕ uniquely.) The map ϕ is "the multiplication" by $B/x = A/y \in Q(R)$, where $Q(R)$ denotes the total quotient ring of R. We have

$$S = k[x, y, z_1, z_2, \dots, z_p, T]/(Ax - By, Tx - B, Ty - A)$$

REMARK S is against Gorenstein (Kustin and Miller, Reid and P.). In general, S is more complicated than both R and R/I. For example, if R is complete intersection codimension 2 and R/I is complete intersection codimension 3 then typically S is not a complete intersection, but a 5×5 Pfaffian.

QUESTION: Assume σ is a face of the simplicial complex D. It is easy to write down generators and relations for $k[D_\sigma]$. But how are $k[D]$ and $k[D_\sigma]$ related in a structural way?

ANSWER (Boehm-P.) If $D \in \mathcal{A}_{\boldsymbol{n}}$ (or more generally D is Gorenstein*) then there is a general way (a kind of tautological construction) to pass from $k[D]$ to $k[D_\sigma]$ via a Kustin–Miller unprojection!

IN MORE DETAIL. Assume the vertex set of D is $\{1, 2, \ldots, m\}$. Without loss of generality assume $\sigma = \{1, 2, \ldots, q\}$. Consider the Stanley–Reisner ring

$$k[D] = k[x_1, \ldots, x_m]/I_D$$

let z be a new variable, and set $R = k[D][z]$ and $x_\sigma = \prod_{i=1}^{q} x_i \in R$.

Define

$$J_\sigma = (0_R : x_\sigma) \subset R$$

that is,

$$J_\sigma = \{f \in R : f x_\sigma = 0\}.$$

Also set

$$I = (J_\sigma, z) \subset R.$$

THEOREM (Boehm-P., 2007) Assume $D \in \mathcal{A}_n$ (or more generally that D is Gorenstein*) . We have that I is a codimension 1 ideal of R. Denote by S the Kustin–Miller unprojection of the pair $I \subset R$. Then z is a regular element of S (i.e., the multiplication by z map $S \to S$ is injective), and we have a (natural) isomorphism

$$S/(z) \to k[D_\sigma]$$

EXAMPLE: Assume D is the (boundary of) 3-gon with vertex set $\{1, 2, 3\}$ and we do a stellar subdivision on the 1-face $\sigma = \{1, 2\}$ of D to get the D_σ, which is the 4-gon. Denote by T (instead of x_4) the variable corresponding to the new vertex 4. We have

$$k[D] = k[x_1, \dots, x_3]/(x_1x_2x_3),$$

$$k[D_\sigma] = k[x_1, \dots, x_3, T]/(x_1x_2, Tx_3)$$

and

$$S = k[x_1, \dots, x_3, z, T]/(Tz - x_1x_2, Tx_3).$$