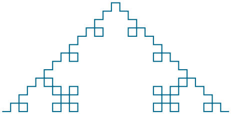
Densities and uniformly distributed measures in the Heisenberg group

Vasilis Chousionis
University of Connecticut



May 26, 2016

Densities of measures

Let μ be a Radon measure on a metric space (X, d). The upper and lower s-densities of μ at $x \in X$ are

$$\Theta^{*s}(\mu, x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s} \text{ and } \Theta_*^s(\mu, x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{r^s}.$$

If the limit exists, call it the s-density of μ at x and denote it by $\Theta^s(\mu, x)$.

Densities of measures

Let μ be a Radon measure on a metric space (X, d). The upper and lower s-densities of μ at $x \in X$ are

$$\Theta^{*s}(\mu, x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s} \text{ and } \Theta^s_*(\mu, x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{r^s}.$$

If the limit exists, call it the s-density of μ at x and denote it by $\Theta^s(\mu, x)$.

- **Lebesgue density theorem :** $A \subset \mathbb{R}^n$, $\mathcal{L}^n$-measurable $\implies$

$$\Theta^n(\mathcal{L}^n \llcorner A, \cdot) = \begin{cases} \mathcal{L}^n(B(0,1)), & \mathcal{L}^n - \text{a.e. in } A \\ 0 & \mathcal{L}^n - \text{a.e. in } \mathbb{R}^n \setminus A \end{cases}$$

s-dimensional Hausdorff measures $\mathcal{H}^s$

Let $A \subset (X, d)$ be any set. Let $s \geq 0$ be any nonnegative real number

1. For any $\delta > 0$ cover A by sets $E_1, E_2, \ldots$ of diameter $\leq \delta$

s-dimensional Hausdorff measures $\mathcal{H}^s$

Let $A \subset (X, d)$ be any set. Let $s \geq 0$ be any nonnegative real number

1. For any $\delta > 0$ cover A by sets $E_1, E_2, \ldots$ of diameter $\leq \delta$
2. Weight each set in the cover by its diameter to power s

s-dimensional Hausdorff measures $\mathcal{H}^s$

Let $A \subset (X, d)$ be any set. Let $s \geq 0$ be any nonnegative real number

1. For any $\delta > 0$ cover A by sets $E_1, E_2, \ldots$ of diameter $\leq \delta$
2. Weight each set in the cover by its diameter to power s
3. Optimize over all such covers

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s : A \subset \bigcup_{i=1}^{\infty} E_i ; \operatorname{diam} E_i \leq \delta \right\}$$

4. Use only finer and finer covers

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

s-dimensional Hausdorff measures $\mathcal{H}^s$

Let $A \subset (X, d)$ be any set. Let $s \geq 0$ be any nonnegative real number

1. For any $\delta > 0$ cover A by sets $E_1, E_2, \ldots$ of diameter $\leq \delta$
2. Weight each set in the cover by its diameter to power s
3. Optimize over all such covers

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s : A \subset \bigcup_{i=1}^{\infty} E_i ; \operatorname{diam} E_i \leq \delta \right\}$$

4. Use only finer and finer covers

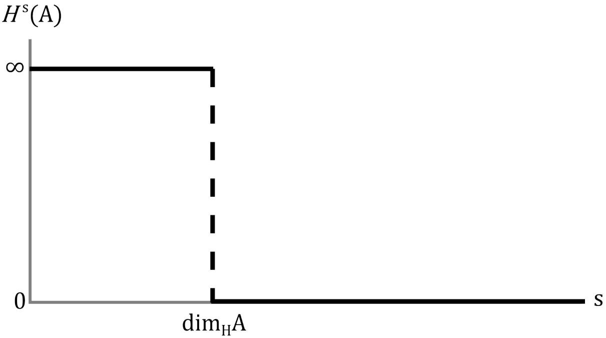
$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

$\mathcal{H}^s$ is called s-dimensional Hausdorff measure

Hausdorff dimension

For any set $A \subset (X, d)$, there is a unique number $s \geq 0$ such that

1. $\mathcal{H}^s(A) = \infty$ for all $s < d$
2. $\mathcal{H}^s(A) = 0$ for all $s > d$

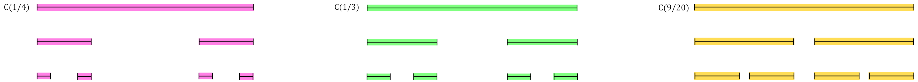


The number $s = \dim_H(A)$ where the transition happens is called the Hausdorff dimension of A.

Hausdorff dimension of Cantor sets

For all $\lambda \in (0, 1/2)$, the Cantor set $C(\lambda)$ has Hausdorff dimension

$$\dim_H C(\lambda) = \frac{\log(2)}{\log(1/\lambda)} \in (0, 1)$$



- $\dim_H C(1/4) = \log(2)/\log(4) = 0.5000000\ldots$
- $\dim_H C(1/3) = \log(2)/\log(3) = 0.6309292\ldots$
- $\dim_H C(9/20) = \log(2)/\log(20/9) = 0.8680532\ldots$
- $\dim_H C(\lambda) \downarrow 0$ as $\lambda \downarrow 0$
- $\dim_H C(\lambda) \uparrow 1$ as $\lambda \uparrow 1/2$

Densities for Hausdorff measures

- If $A \subset \mathbb{R}^n$, $\mathcal{H}^s$-measurable and $\mathcal{H}^s(A) < \infty$,

$$1 \leq \Theta^{*s}(\mathcal{H}^s \llcorner A, \cdot) \leq 2^s, \quad \mathcal{H}^s\text{-a.e. in } A$$

Densities for Hausdorff measures

- If $A \subset \mathbb{R}^n$, $\mathcal{H}^s$-measurable and $\mathcal{H}^s(A) < \infty$,

$$1 \leq \Theta^{*s}(\mathcal{H}^s \llcorner A, \cdot) \leq 2^s, \quad \mathcal{H}^s\text{-a.e. in } A$$

and

$$\Theta^s(\mathcal{H}^s \llcorner A, \cdot) = 0, \mathcal{H}^s\text{-a.e. in } \mathbb{R}^n \setminus A.$$

Marstrand density theorem

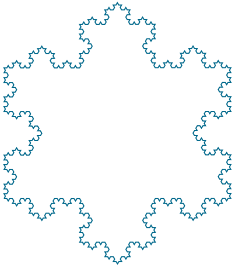
Theorem (Marstrand 1954 and 1964)

Suppose for some $s > 0$*, that there exists a Radon measure* μ *in* $\mathbb{R}^n$ *s.t.* $\Theta^s(\mu, \cdot)$ *exists and is positive and finite in a set of positive* μ*-measure. Then* $s \in \mathbb{N}$*.*

Marstrand density theorem

Theorem (Marstrand 1954 and 1964)

Suppose for some $s > 0$, that there exists a Radon measure μ in $\mathbb{R}^n$ s.t. $\Theta^s(\mu, \cdot)$ exists and is positive and finite in a set of positive μ-measure. Then $s \in \mathbb{N}$.



Sets with noninteger Hausdorff dimension are irregular at 'generic' points.

Preiss density theorem

Theorem (Preiss 1987)

Let μ be a Radon measure in $\mathbb{R}^n$ such that the density $\Theta^m(\mu, \cdot)$, $m \in \mathbb{N}$, exists and is positive and finite μ a.e. Then $\operatorname{supp}(\mu)$ *is m-rectifiable and*

$\mu \ll \mathcal{H}^m \llcorner \operatorname{supp}(\mu)$.

$E \subset \mathbb{R}^n$ is *m-rectifiable* if there exist countably many m-dimensional Lipschitz graphs M_i such that

$$\mathcal{H}^m(E \setminus \cup M_i) = 0.$$

Preiss density theorem

Theorem (Preiss 1987)

Let μ be a Radon measure in $\mathbb{R}^n$ such that the density $\Theta^m(\mu, \cdot), m \in \mathbb{N}$, exists and is positive and finite μ a.e. Then $\mathrm{supp}(\mu)$ *is m-rectifiable and*

$\mu \ll \mathcal{H}^m \llcorner \mathrm{supp}(\mu)$.

$E \subset \mathbb{R}^n$ is *m-rectifiable* if there exist countably many m-dimensional Lipschitz graphs M_i such that

$$\mathcal{H}^m(E \setminus \cup M_i) = 0.$$

Eariler work by Besicovitch, Morse-Randolph, Moore.

A Marstrand-type theorems via Wolff potentials

Theorem (C, Prat, Tolsa)

Let μ be a Radon measure in $\mathbb{R}^n$ and let $s \notin \mathbb{Z}$. Then

$$\int \int_0^\infty \left(\frac{\mu(B(x,r))}{r^s} - \frac{\mu(B(x,2r))}{(2r)^s} \right)^2 \frac{dr}{r}\, d\mu(x) \approx \int \int_0^\infty \left(\frac{\mu(B(x,r))}{r^s} \right)^2 \frac{dr}{r}\, d\mu(x).$$

A Marstrand-type theorems via Wolff potentials

Theorem (C, Prat, Tolsa)

Let μ be a Radon measure in $\mathbb{R}^n$ and let $s \notin \mathbb{Z}$. Then

$$\int \int_0^\infty \left(\frac{\mu(B(x,r))}{r^s} - \frac{\mu(B(x,2r))}{(2r)^s} \right)^2 \frac{dr}{r}\, d\mu(x) \approx \int \int_0^\infty \left(\frac{\mu(B(x,r))}{r^s} \right)^2 \frac{dr}{r}\, d\mu(x).$$

- The theorem fails for $n \in \mathbb{N}$.

A Marstrand-type theorems via Wolff potentials

Theorem (C, Prat, Tolsa)

Let μ be a Radon measure in $\mathbb{R}^n$ and let $s \notin \mathbb{Z}$. Then

$$\int \int_0^\infty \left(\frac{\mu(B(x,r))}{r^s} - \frac{\mu(B(x,2r))}{(2r)^s} \right)^2 \frac{dr}{r} \, d\mu(x) \approx \int \int_0^\infty \left(\frac{\mu(B(x,r))}{r^s} \right)^2 \frac{dr}{r} \, d\mu(x).$$

- The theorem fails for $n \in \mathbb{N}$.
- Wolff potentials, $\int_0^\infty \frac{\mu(B(x,r))^2}{r^{2s}} \frac{dr}{r}$ used to characterize Sobolev spaces.

A Marstrand-type theorems via Wolff potentials

Theorem (C, Prat, Tolsa)

Let μ be a Radon measure in $\mathbb{R}^n$ and let $s \notin \mathbb{Z}$. Then

$$\int \int_0^\infty \left(\frac{\mu(B(x,r))}{r^s} - \frac{\mu(B(x,2r))}{(2r)^s} \right)^2 \frac{dr}{r}\, d\mu(x) \approx \int \int_0^\infty \left(\frac{\mu(B(x,r))}{r^s} \right)^2 \frac{dr}{r}\, d\mu(x).$$

- The theorem fails for $n \in \mathbb{N}$.
- Wolff potentials, $\int_0^\infty \frac{\mu(B(x,r))^2}{r^{2s}} \frac{dr}{r}$ used to characterize Sobolev spaces.
- (C, Garnett, Le, Tolsa) introduced $\Delta_\mu^s(x,r) = \frac{\mu(B(x,r))}{r^s} - \frac{\mu(B(x,2r))}{(2r)^s}$ to give a characterization of **uniform rectifiability** using densities.

A lemma from the proof

Lemma (C, Prat, Tolsa)

Let μ *be a nontrivial Radon measure in* $\mathbb{R}^n$ *and let* $s \notin \mathbb{Z}$. *Then there exist* $x_0 \in \operatorname{supp}(\mu)$ *and* $r_0 > 0$ *such that* $\Delta_\mu^s(x_0, r_0) \neq 0$, *that is*

$$\frac{\mu(B(x_0, r_0))}{r_0^s} \neq \frac{\mu(B(x_0, 2r_0))}{(2r_0)^s}.$$

A lemma from the proof

Lemma (C, Prat, Tolsa)

Let μ be a nontrivial Radon measure in $\mathbb{R}^n$ and let $s \notin \mathbb{Z}$. Then there exist $x_0 \in \operatorname{supp}(\mu)$ and $r_0 > 0$ such that $\Delta_\mu^s(x_0, r_0) \neq 0$, that is

$$\frac{\mu(B(x_0, r_0))}{r_0^s} \neq \frac{\mu(B(x_0, 2r_0))}{(2r_0)^s}.$$

- The proof depends on the Euclidean metric.

A lemma from the proof

Lemma (C, Prat, Tolsa)

Let μ be a nontrivial Radon measure in $\mathbb{R}^n$ and let $s \notin \mathbb{Z}$. Then there exist $x_0 \in \operatorname{supp}(\mu)$ and $r_0 > 0$ such that $\Delta^s_\mu(x_0, r_0) \neq 0$, that is

$$\frac{\mu(B(x_0, r_0))}{r_0^s} \neq \frac{\mu(B(x_0, 2r_0))}{(2r_0)^s}.$$

- The proof depends on the Euclidean metric.
- (C, Rajala) True for $s < 1$ even when (X, d) is a complete metric space.

A lemma from the proof

Lemma (C, Prat, Tolsa)

Let μ be a nontrivial Radon measure in $\mathbb{R}^n$ and let $s \notin \mathbb{Z}$. Then there exist $x_0 \in \operatorname{supp}(\mu)$ and $r_0 > 0$ such that $\Delta_\mu^s(x_0, r_0) \neq 0$, that is

$$\frac{\mu(B(x_0, r_0))}{r_0^s} \neq \frac{\mu(B(x_0, 2r_0))}{(2r_0)^s}.$$

- The proof depends on the Euclidean metric.
- (C, Rajala) True for $s < 1$ even when (X, d) is a complete metric space.
- Unknown for other metrics in $\mathbb{R}^n$.

Results for other metric spaces

Several papers by A. Lorent about metrics in $\mathbb{R}^n$ defined by centrally symmetric convex polytopes. For example, he proves

- Marstrand's theorem for d_∞ and $s \leq 2$.
- locally 2-uniform measures in $(\mathbb{R}^3, d_\infty)$ are rectifiable.

The Heisenberg group

- $\mathbb{H}^n = \mathbb{R}^{2n+1} \ni p = (z,t) = (x,y,t) = (x_1,\dots,x_n,y_1,\dots,y_n,t)$
- $(z,t) * (z',t') = (z+z', t+t' + 2\omega(z,z'))$,
  where $\omega(z,z') = x \cdot y' - x' \cdot y$
- **Horizontal distribution:**

$$H_p\mathbb{H}^n = \operatorname{span}\{X_1(p), Y_1(p), \dots, X_n(p), Y_n(p)\}$$

$$X_1 = \frac{\partial}{\partial x_1} + 2y_1\frac{\partial}{\partial t}, \dots, X_n = \frac{\partial}{\partial x_n} + 2y_n\frac{\partial}{\partial t}$$
$$Y_1 = \frac{\partial}{\partial y_1} - 2x_1\frac{\partial}{\partial t}, \dots, Y_n = \frac{\partial}{\partial y_n} - 2x_n\frac{\partial}{\partial t}$$

- **Gauge (Korányi) metric:**

$$d_{\mathbb{H}}(p,q) = \|p^{-1} * q\|_{\mathbb{H}}, \quad \|(z,t)\|_{\mathbb{H}} = (|z|^4 + t^2)^{1/4}$$

- **Dilations:** $\delta_r : \mathbb{H}^n \to \mathbb{H}^n, r > 0$,

$$\delta_r(z,t) = (rz, r^2 t)$$

The geometry of $\mathbb{H}$: Sub-Riemannian structure

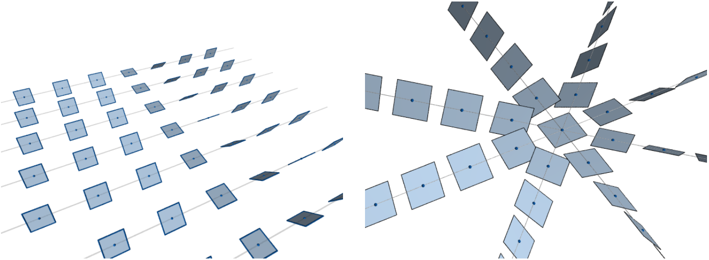
Let X_1, X_2 be the left invariant vector fields

$$X_1 = \partial_{x_1} + 2x_2\partial_{x_3} \text{ and } X_2 = \partial_{x_2} - 2x_1\partial_{x_3}.$$

The geometry of $\mathbb{H}$: Sub-Riemannian structure

Let X_1, X_2 be the left invariant vector fields

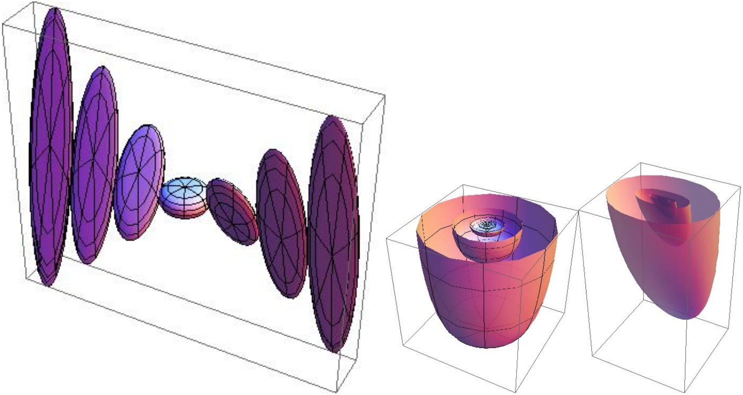
$$X_1 = \partial_{x_1} + 2x_2\partial_{x_3} \text{ and } X_2 = \partial_{x_2} - 2x_1\partial_{x_3}.$$



- Motion is only allowed along the horizontal planes:

$$H_p\mathbb{H}^1 = \text{span}\{X_1(p), X_2(p)\}.$$

Balls in $(\mathbb{H}, d_H)$



$\dim_H(\mathbb{H}) = 4$

Marstrand's theorem in the Heisenberg group

Recall, $d_{\mathbb{H}}(p,q) = \|p^{-1} * q\|_{\mathbb{H}}, \quad \|(z,t)\|_{\mathbb{H}} = (|z|^4 + t^2)^{1/4}$

Marstrand's theorem in the Heisenberg group

Recall, $d_{\mathbb{H}}(p,q) = \|p^{-1} * q\|_{\mathbb{H}}, \quad \|(z,t)\|_{\mathbb{H}} = (|z|^4 + t^2)^{1/4}$

Theorem (C, Tyson)

Let μ be a Radon measure in $(\mathbb{H}^n, d_{\mathbb{H}})$. If the density $\Theta^s_{\mathbb{H}}(\mu, \cdot)$ exists and is positive and finite in a set of positive μ-measure, then $s \in \mathbb{N}$.

Marstrand's theorem in the Heisenberg group

Recall, $d_{\mathbb{H}}(p,q) = \|p^{-1} * q\|_{\mathbb{H}}, \quad \|(z,t)\|_{\mathbb{H}} = (|z|^4 + t^2)^{1/4}$

Theorem (C, Tyson)

Let μ be a Radon measure in $(\mathbb{H}^n, d_{\mathbb{H}})$. If the density $\Theta^s_{\mathbb{H}}(\mu, \cdot)$ exists and is positive and finite in a set of positive μ-measure, then $s \in \mathbb{N}$.

Remarks

- Marstrand's proof is very Euclidean.

Marstrand's theorem in the Heisenberg group

Recall, $d_{\mathbb{H}}(p,q) = \|p^{-1} * q\|_{\mathbb{H}}, \quad \|(z,t)\|_{\mathbb{H}} = (|z|^4 + t^2)^{1/4}$

Theorem (C, Tyson)

Let μ be a Radon measure in $(\mathbb{H}^n, d_{\mathbb{H}})$. If the density $\Theta^s_{\mathbb{H}}(\mu,\cdot)$ exists and is positive and finite in a set of positive μ-measure, then $s \in \mathbb{N}$.

Remarks

- Marstrand's proof is very Euclidean.
- Another proof, due to Kirchheim and Preiss, uses *uniformly distributed measures*.

Uniformly distributed and uniform measures

Definition

A Radon measure on a metric space (X, d) is called uniformly distributed if

$$\mu(B(x,r)) = \mu(B(y,r))$$

for all $x, y \in \operatorname{supp}(\mu)$ and all $r > 0$.

Uniformly distributed and uniform measures

Definition

A Radon measure on a metric space (X, d) is called uniformly distributed if

$$\mu(B(x,r)) = \mu(B(y,r))$$

for all $x, y \in \mathrm{supp}(\mu)$ and all $r > 0$.

Definition

Let $s > 0$. A Radon measure on a metric space (X, d) is called s-uniform if there exists some constant $c > 0$ such that

$$\mu(B(x,r)) = cr^s$$

for all $x \in \mathrm{supp}(\mu)$ and all $r > 0$.

Kirchheim-Preiss Theorem

Theorem (Kirchheim-Preiss)

Let μ be a uniformly distributed measure in $\mathbb{R}^n$. Then $\operatorname{supp}(\mu)$ is a real analytic variety.

Kirchheim-Preiss Theorem

Theorem (Kirchheim-Preiss)

Let μ be a uniformly distributed measure in $\mathbb{R}^n$. Then $\operatorname{supp}(\mu)$ is a real analytic variety.

Theorem (C, Tyson)

Let μ be a uniformly distributed measure in $(\mathbb{H}^n, d_{\mathbb{H}})$. Then $\operatorname{supp}(\mu)$ is a real analytic variety in $\mathbb{R}^{2n+1}$.

Tangent measures and uniform measures in Marstrand's theorem

Let $\mathbb{G}$ be $\mathbb{R}^n$ or $\mathbb{H}^n$. Let μ be a Radon measure on $\mathbb{G}$. For $a \in \mathbb{G}, r > 0$, denote the blow-ups of μ from $B(a, r)$ to $B(0, 1)$ by

$$\mu_{a,r}(A) = \mu(a * (\delta_r(A)) \, , A \subset \mathbb{G}.$$

Tangent measures and uniform measures in Marstrand's theorem

Let $\mathbb{G}$ be $\mathbb{R}^n$ or $\mathbb{H}^n$. Let μ be a Radon measure on $\mathbb{G}$. For $a \in \mathbb{G}, r > 0$, denote the blow-ups of μ from $B(a, r)$ to $B(0, 1)$ by

$$\mu_{a,r}(A) = \mu(a * (\delta_r(A)) \, , A \subset \mathbb{G}.$$

ν is a **tangent measure** of μ at $a \in \mathbb{G}$ if ν is a Radon measure on $\mathbb{G}$ with $\nu(\mathbb{G}) > 0$ and there are $c_i > 0$ and $r_i > 0$, $i \in \mathbb{N}$, such that $r_i \to 0$ and

$$c_i \mu_{a,r_i} \to \nu \text{ weakly as } i \to \infty.$$

Tangent measures and uniform measures in Marstrand's theorem

Let $\mathbb{G}$ be $\mathbb{R}^n$ or $\mathbb{H}^n$. Let μ be a Radon measure on $\mathbb{G}$. For $a \in \mathbb{G}, r > 0$, denote the blow-ups of μ from $B(a,r)$ to $B(0,1)$ by

$$\mu_{a,r}(A) = \mu(a * (\delta_r(A)) \, , A \subset \mathbb{G}.$$

ν is a **tangent measure** of μ at $a \in \mathbb{G}$ if ν is a Radon measure on $\mathbb{G}$ with $\nu(\mathbb{G}) > 0$ and there are $c_i > 0$ and $r_i > 0$, $i \in \mathbb{N}$, such that $r_i \to 0$ and

$$c_i \mu_{a,r_i} \to \nu \text{ weakly as } i \to \infty.$$

We denote by $\mathrm{Tan}(\mu, a)$ the set of all tangent measures of μ at a.

Tangent measures and uniform measures in Marstrand's theorem

Let $\mathbb{G}$ be $\mathbb{R}^n$ or $\mathbb{H}^n$. Let μ be a Radon measure on $\mathbb{G}$. For $a \in \mathbb{G}, r > 0$, denote the blow-ups of μ from $B(a, r)$ to $B(0, 1)$ by

$$\mu_{a,r}(A) = \mu(a * (\delta_r(A)))\ , A \subset \mathbb{G}.$$

ν is a **tangent measure** of μ at $a \in \mathbb{G}$ if ν is a Radon measure on $\mathbb{G}$ with $\nu(\mathbb{G}) > 0$ and there are $c_i > 0$ and $r_i > 0$, $i \in \mathbb{N}$, such that $r_i \to 0$ and

$$c_i \mu_{a,r_i} \to \nu \text{ weakly as } i \to \infty.$$

We denote by $\mathrm{Tan}(\mu, a)$ the set of all tangent measures of μ at a.

Proposition

Let $s > 0$. Let μ be a Radon measure such that $0 < \Theta^s(\mu, \cdot) < \infty$ exists μ-a.e. in $A \subset \mathbb{G}$. Then for μ a.e. $a \in A$, $\mathrm{Tan}(\mu, a)$, consists of s-uniform measures.

Marstrand theorem: scheme of the proof

$$0 < \Theta^s(\mu, \cdot) < \infty \text{ exists } \mu - \text{ a.e. in } A \subset \mathbb{H}^n$$

Marstrand theorem: scheme of the proof

$$0 < \Theta^s(\mu, \cdot) < \infty \text{ exists } \mu - \text{ a.e. in } A \subset \mathbb{H}^n$$

$$\Downarrow$$

$$\exists \quad s - \text{uniform measure } \nu$$

Marstrand theorem: scheme of the proof

$$0 < \Theta^s(\mu, \cdot) < \infty \text{ exists } \mu - \text{ a.e. in } A \subset \mathbb{H}^n$$

$$\Downarrow$$

$$\exists \quad s - \text{uniform measure } \nu$$

$$\Downarrow \qquad \text{(Kirhchheim-Preiss Theorem)}$$

$$\text{supp}(\nu) \text{ is a real analytic variety}$$

Marstrand theorem: scheme of the proof

$$0 < \Theta^s(\mu, \cdot) < \infty \text{ exists } \mu - \text{ a.e. in } A \subset \mathbb{H}^n$$

$$\Downarrow$$

$$\exists \quad s - \text{uniform measure } \nu$$

$$\Downarrow \qquad \text{(Kirhchheim-Preiss Theorem)}$$

$$\text{supp}(\nu) \text{ is a real analytic variety}$$

$$\Downarrow \qquad \text{(Lojasiewicz Theorem)}$$

$$\text{supp}(\nu) \text{ is a countable union of real analytic submanifolds}$$

Marstrand theorem: scheme of the proof

$$0 < \Theta^s(\mu, \cdot) < \infty \text{ exists } \mu - \text{ a.e. in } A \subset \mathbb{H}^n$$

$$\Downarrow$$

$$\exists \quad s - \text{uniform measure } \nu$$

$$\Downarrow \qquad \text{(Kirhchheim-Preiss Theorem)}$$

$$\text{supp}(\nu) \text{ is a real analytic variety}$$

$$\Downarrow \qquad \text{(Lojasiewicz Theorem)}$$

$$\text{supp}(\nu) \text{ is a countable union of real analytic submanifolds}$$

$$\Downarrow \qquad \text{(Gromov Theorem)}$$

$$\dim_H(\text{supp}(\nu)) \in \mathbb{N}$$

Marstrand theorem: scheme of the proof

$$0 < \Theta^s(\mu, \cdot) < \infty \text{ exists } \mu - \text{ a.e. in } A \subset \mathbb{H}^n$$

$$\Downarrow$$

$$\exists \quad s - \text{uniform measure } \nu$$

$$\Downarrow \quad \text{(Kirhchheim-Preiss Theorem)}$$

$$\operatorname{supp}(\nu) \text{ is a real analytic variety}$$

$$\Downarrow \quad \text{(Lojasiewicz Theorem)}$$

$$\operatorname{supp}(\nu) \text{ is a countable union of real analytic submanifolds}$$

$$\Downarrow \quad \text{(Gromov Theorem)}$$

$$\dim_H(\operatorname{supp}(\nu)) \in \mathbb{N}$$

$$\Downarrow$$

$$s \in \mathbb{N}.$$

Hausdorff dimension of submanifolds of $\mathbb{H}^n$

Theorem (Gromov)

Let Σ be an m-dimensional $C^{1,1}$ submanifold in $\mathbb{H}^n$, then

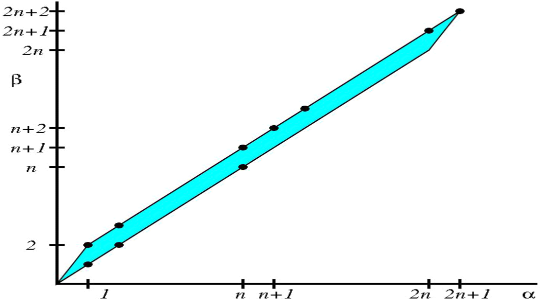
$$\dim_H(\Sigma) = \bar{m} \in \{m, m+1\}.$$

Hausdorff dimension of submanifolds of $\mathbb{H}^n$

Theorem (Gromov)

Let Σ be an m-dimensional $C^{1,1}$ submanifold in $\mathbb{H}^n$, then

$$\dim_H(\Sigma) = \bar{m} \in \{m, m+1\}.$$



Wolff potentials in $(\mathbb{H}^n, d_{\mathbb{H}})$

Lemma (C, Tyson)

Let μ be a nontrivial Radon measure in $(\mathbb{H}^n, d_{\mathbb{H}})$ and let $s \notin \mathbb{Z}$. Then there exist $x_0 \in \operatorname{supp}(\mu)$ and $r_0 > 0$ such that

$$\frac{\mu(B(x_0, r_0))}{r_0^s} \neq \frac{\mu(B(x_0, 2r_0))}{(2r_0)^s}.$$

Wolff potentials in $(\mathbb{H}^n, d_{\mathbb{H}})$

Lemma (C, Tyson)

Let μ be a nontrivial Radon measure in $(\mathbb{H}^n, d_{\mathbb{H}})$ and let $s \notin \mathbb{Z}$. Then there exist $x_0 \in \operatorname{supp}(\mu)$ and $r_0 > 0$ such that

$$\frac{\mu(B(x_0, r_0))}{r_0^s} \neq \frac{\mu(B(x_0, 2r_0))}{(2r_0)^s}.$$

$$\Downarrow$$

Theorem (C, Tyson)

Let μ be a Radon measure in $(\mathbb{H}^n, d_{\mathbb{H}})$ and let $s \notin \mathbb{Z}$. Then

$$\int \int_0^\infty \left(\frac{\mu(B(x,r))}{r^s} - \frac{\mu(B(x,2r))}{(2r)^s} \right)^2 \frac{dr}{r} \, d\mu(x) \approx \int \int_0^\infty \frac{\mu(B(x,r))^2}{r^{2s}} \frac{dr}{r} \, d\mu(x).$$

Uniformly distributed measures with bdd support

- (Kirchheim-Preiss) Bounded supports of Euclidean uniformly distributed measures are contained in spheres

Uniformly distributed measures with bdd support

- (Kirchheim-Preiss) Bounded supports of Euclidean uniformly distributed measures are contained in spheres
- (C, Tyson) Bounded supports of uniformly distributed measures on $\mathbb{H}^n$ are algebraic varieties

Uniformly distributed measures with bdd support

- (Kirchheim-Preiss) Bounded supports of Euclidean uniformly distributed measures are contained in spheres
- (C, Tyson) Bounded supports of uniformly distributed measures on $\mathbb{H}^n$ are algebraic varieties
- (Kirchheim-Preiss) Uniformly distributed counting measures with **finite** support in $\mathbb{R}^2$ are supported on either the vertices of a regular polygon, or two regular m-gons lying on a common circle

Uniformly distributed measures with bdd support

- (Kirchheim-Preiss) Bounded supports of Euclidean uniformly distributed measures are contained in spheres
- (C, Tyson) Bounded supports of uniformly distributed measures on $\mathbb{H}^n$ are algebraic varieties
- (Kirchheim-Preiss) Uniformly distributed counting measures with **finite** support in $\mathbb{R}^2$ are supported on either the vertices of a regular polygon, or two regular m-gons lying on a common circle
- Characterization of uniformly distributed counting measures with **finite** support in $\mathbb{H}^1$ is open.

Uniformly distributed measures with finite support

A set A in a metric space (X, d) is called *equilateral* if $d(x, y)$ is constant for all $x, y \in A$, $x \neq y$.

- Counting measure is uniformly distributed on each equilateral set

Uniformly distributed measures with finite support

A set A in a metric space (X, d) is called *equilateral* if $d(x, y)$ is constant for all $x, y \in A$, $x \neq y$.

- Counting measure is uniformly distributed on each equilateral set
- (C, Tyson) Characterization equilateral triangles in $(\mathbb{H}, d_{\mathbb{H}})$. Such triangles fall into three distinct classes:
  1. two vertices lie on a vertical line,
  2. two vertices lie on a horizontal line,
  3. no two vertices lie on either a horizontal or a vertical line.

Uniformly distributed measures with finite support

A set A in a metric space (X, d) is called *equilateral* if $d(x, y)$ is constant for all $x, y \in A$, $x \neq y$.

- Counting measure is uniformly distributed on each equilateral set
- (C, Tyson) Characterization equilateral triangles in $(\mathbb{H}, d_{\mathbb{H}})$. Such triangles fall into three distinct classes:
  1. two vertices lie on a vertical line,
  2. two vertices lie on a horizontal line,
  3. no two vertices lie on either a horizontal or a vertical line.
- We declined to characterize equilateral 4 point sets (probably tractable)

Uniformly distributed measures with finite support

A set A in a metric space (X, d) is called *equilateral* if $d(x, y)$ is constant for all $x, y \in A$, $x \neq y$.

- Counting measure is uniformly distributed on each equilateral set
- (C, Tyson) Characterization equilateral triangles in $(\mathbb{H}, d_{\mathbb{H}})$. Such triangles fall into three distinct classes:
  1. two vertices lie on a vertical line,
  2. two vertices lie on a horizontal line,
  3. no two vertices lie on either a horizontal or a vertical line.
- We declined to characterize equilateral 4 point sets (probably tractable)
- Unknown if there exist equilateral 5 point subsets of $(\mathbb{H}^1, d_{\mathbb{H}})$!

Uniform measures in $(\mathbb{R}^n, d_E)$

- (Preiss) For $m = 1, 2$ all m-uniform measures are m-flat; a measure μ is m-flat if $\mu = c_\mu \mathcal{H}^m \llcorner V$, $V \in G(n, m)$.

Uniform measures in $(\mathbb{R}^n, d_E)$

- (Preiss) For $m = 1, 2$ all m-uniform measures are m-flat; a measure μ is m-flat if $\mu = c_\mu \mathcal{H}^m \llcorner V, V \in G(n, m)$.
- (Preiss) The light cone $C = \{x \in \mathbb{R}^4 : x_1^2 = x_2^2 + x_3^2 + x_4^2\}$ is 3-uniform!

Uniform measures in $(\mathbb{R}^n, d_E)$

- (Preiss) For $m = 1, 2$ all m-uniform measures are m-flat; a measure μ is m-flat if $\mu = c_\mu \mathcal{H}^m \llcorner V, V \in G(n, m)$.
- (Preiss) The light cone $C = \{x \in \mathbb{R}^4 : x_1^2 = x_2^2 + x_3^2 + x_4^2\}$ is 3-uniform!
- (Preiss-Kowalski) Every $(n-1)$-uniform measure in $\mathbb{R}^n$ is either $(n-1)$-flat or is a constant multiple of $\mathcal{H}^{n-1}$ on some isometric copy of $C \times \mathbb{R}^{n-4}$.

Uniform measures in $(\mathbb{R}^n, d_E)$

- (Preiss) For $m = 1, 2$ all m-uniform measures are m-flat; a measure μ is m-flat if $\mu = c_\mu \mathcal{H}^m \llcorner V, V \in G(n, m)$.
- (Preiss) The light cone $C = \{x \in \mathbb{R}^4 : x_1^2 = x_2^2 + x_3^2 + x_4^2\}$ is 3-uniform!
- (Preiss-Kowalski) Every $(n-1)$-uniform measure in $\mathbb{R}^n$ is either $(n-1)$-flat or is a constant multiple of $\mathcal{H}^{n-1}$ on some isometric copy of $C \times \mathbb{R}^{n-4}$.
- Classification open for $3 \leq m < n - 1$.

Classification of uniform measures in $(\mathbb{H}^1, d_{\mathbb{H}})$

Ongoing project with V. Magnani and J. Tyson.

Marstrand's theorem in $\mathbb{H}^n \implies$ there are no s-uniform measures for $s \notin \mathbb{N}$.

Classification of uniform measures in $(\mathbb{H}^1, d_{\mathbb{H}})$

Ongoing project with V. Magnani and J. Tyson.

Marstrand's theorem in $\mathbb{H}^n \implies$ there are no s-uniform measures for $s \notin \mathbb{N}$.

Conjecture

Let μ be a $\overline{m}$-uniform measure on $\mathbb{H}$. Then

- *if $\overline{m} = 1$, then $\mu = c_\mu \mathcal{H}^1 \llcorner L$ for some horizontal line L,*
- *if $\overline{m} = 2$, then $\mu = c_\mu \mathcal{H}^2 \llcorner V$ for some vertical line V,*
- *if $\overline{m} = 3$, then $\mu = c_\mu \mathcal{H}^3 \llcorner W$ for some vertical plane W.*

Classification of uniform measures in $(\mathbb{H}^1, d_{\mathbb{H}})$

Ongoing project with V. Magnani and J. Tyson.

Marstrand's theorem in $\mathbb{H}^n \implies$ there are no s-uniform measures for $s \notin \mathbb{N}$.

Conjecture

Let μ be a $\overline{m}$-uniform measure on $\mathbb{H}$. Then

- *if $\overline{m} = 1$, then $\mu = c_\mu \mathcal{H}^1 \llcorner L$ for some horizontal line L,* TRUE

Classification of uniform measures in $(\mathbb{H}^1, d_{\mathbb{H}})$

Ongoing project with V. Magnani and J. Tyson.

Marstrand's theorem in $\mathbb{H}^n \implies$ there are no s-uniform measures for $s \notin \mathbb{N}$.

Conjecture

Let μ be a $\overline{m}$-uniform measure on $\mathbb{H}$. Then

- *if $\overline{m} = 1$, then $\mu = c_\mu \mathcal{H}^1 \llcorner L$ for some horizontal line L,* TRUE
- *if $\overline{m} = 2$, then $\mu = c_\mu \mathcal{H}^2 \llcorner V$ for some vertical line V, when $\operatorname{supp}(\mu)$ is contained in a vertical plane*

Classification of uniform measures in $(\mathbb{H}^1, d_{\mathbb{H}})$

Ongoing project with V. Magnani and J. Tyson.

Marstrand's theorem in $\mathbb{H}^n \implies$ there are no s-uniform measures for $s \notin \mathbb{N}$.

Conjecture

Let μ be a $\overline{m}$-uniform measure on $\mathbb{H}$. Then

- *if $\overline{m} = 1$, then $\mu = c_\mu \mathcal{H}^1 \llcorner L$ for some horizontal line L,* *TRUE*
- *if $\overline{m} = 2$, then $\mu = c_\mu \mathcal{H}^2 \llcorner V$ for some vertical line V, when $\mathrm{supp}(\mu)$ is contained in a vertical plane*
- *if $\overline{m} = 3$, then $\mu = c_\mu \mathcal{H}^3 \llcorner W$ for some vertical plane W when $\mathrm{supp}(\mu)$ is contained in a vertically ruled surface*

Classification of uniform measures in $(\mathbb{H}^1, d_{\mathbb{H}})$

Denote

$$\Sigma = \Sigma_{(0)} \cup \cdots \cup \Sigma_{(m)},$$

the stratification of the support $\Sigma = \mathrm{supp}(\mu)$ of a uniform measure μ into analytic submanifolds of dimensions between 0 and m.

Theorem

Let μ be a 2-uniform measure on $\mathbb{H}$. Then

(i) $\Sigma_{(1)}$ *is a fully nonhorizontal curve.*

(ii) *the following geometric PDE is satisfied at all points of* $\Sigma_{(1)}$:

$$k_\gamma \circ \pi = \frac{3}{2} k^0_\Sigma,$$

for $\gamma = \pi \circ \Sigma$

$k_\gamma =$ the usual curvature of a plane curve

$k^0_\Sigma =$ the intrinsic curvature of the (nonhorizontal) curve $\Sigma \subset \mathbb{H}^1$

Classification of uniform measures in $(\mathbb{H}^1, d_{\mathbb{H}})$

Theorem

Let μ be a 2-uniform measure on $\mathbb{H}$. Then

(i) $\Sigma_{(1)}$ *is a fully nonhorizontal curve.*

(ii) *the following geometric PDE is satisfied at all points of* $\Sigma_{(1)}$*:*

$$\frac{2}{3}\tau(\dot{\Sigma})(\dot{x}\ddot{y} - \dot{y}\ddot{x}) = |\dot{\gamma}|^4$$

for $\gamma = \pi \circ \Sigma$.

Classification of uniform measures in $(\mathbb{H}^1, d_{\mathbb{H}})$

Theorem

Let μ be a 3-uniform measure on $\mathbb{H}$. Then

(i) *All points of in the top dimensional stratum Σ_2 of Σ are noncharacteristic.*

(ii) *The following geometric PDE is satisfied at all points of Σ_2:*

$$\mathcal{H}_0^2 - 4\mathcal{K}_0 - \frac{5}{2}\mathcal{P}_0^2 = 0.$$

$\mathcal{H}_0 =$ horizontal mean curvature

$\mathcal{P}_0 =$ imaginary curvature

$\mathcal{K}_0 =$ the horizontal Gauss curvature.

Curvatures

- $x \in \Sigma$ is *noncharacteristic* if $T_x\Sigma \neq H_x\mathbb{H}$. At such points, there exists a unique curve $\gamma_x : (-\epsilon, \epsilon) \to \Sigma$, passing through x, s.t. $\gamma_x'(0) \in H_x\mathbb{H} \cap T_x\Sigma =: HT_x\Sigma$.

$$\mathcal{H}_0(x) := \kappa_{\pi\circ\gamma_x}(0),$$

where κ_c denotes the *curvature* of a curve c in $\mathbb{R}^2$.

Curvatures

- $x \in \Sigma$ is *noncharacteristic* if $T_x\Sigma \neq H_x\mathbb{H}$. At such points, there exists a unique curve $\gamma_x : (-\epsilon, \epsilon) \to \Sigma$, passing through x, s.t. $\gamma_x'(0) \in H_x\mathbb{H} \cap T_x\Sigma =: HT_x\Sigma$.

$$\mathcal{H}_0(x) := \kappa_{\pi \circ \gamma_x}(0),$$

  where κ_c denotes the *curvature* of a curve c in $\mathbb{R}^2$.
- The *metric normal* at a noncharacteristic point $x \in \Sigma$, denoted $\mathcal{N}_x\Sigma$, is the set of points $y \in \mathbb{H}$ such that $\mathrm{dist}_{cc}(y, \Sigma) = d_{cc}(y, x)$.

Curvatures

- $x \in \Sigma$ is *noncharacteristic* if $T_x\Sigma \neq H_x\mathbb{H}$. At such points, there exists a unique curve $\gamma_x : (-\epsilon, \epsilon) \to \Sigma$, passing through x, s.t. $\gamma_x'(0) \in H_x\mathbb{H} \cap T_x\Sigma =: HT_x\Sigma$.

$$\mathcal{H}_0(x) := \kappa_{\pi\circ\gamma_x}(0),$$

  where κ_c denotes the *curvature* of a curve c in $\mathbb{R}^2$.
- The *metric normal* at a noncharacteristic point $x \in \Sigma$, denoted $\mathcal{N}_x\Sigma$, is the set of points $y \in \mathbb{H}$ such that $\mathrm{dist}_{cc}(y, \Sigma) = d_{cc}(y, x)$.
- The *imaginary curvature* of Σ at x, denoted $\mathcal{P}_0(x)$, is the horizontal curvature of $\mathcal{N}_x\Sigma$ at x.

Curvatures

- $x \in \Sigma$ is *noncharacteristic* if $T_x\Sigma \neq H_x\mathbb{H}$. At such points, there exists a unique curve $\gamma_x : (-\epsilon, \epsilon) \to \Sigma$, passing through x, s.t. $\gamma_x'(0) \in H_x\mathbb{H} \cap T_x\Sigma =: HT_x\Sigma$.

$$\mathcal{H}_0(x) := \kappa_{\pi \circ \gamma_x}(0),$$

where κ_c denotes the *curvature* of a curve c in $\mathbb{R}^2$.
- The *metric normal* at a noncharacteristic point $x \in \Sigma$, denoted $\mathcal{N}_x\Sigma$, is the set of points $y \in \mathbb{H}$ such that $\mathrm{dist}_{cc}(y, \Sigma) = d_{cc}(y, x)$.
- The *imaginary curvature* of Σ at x, denoted $\mathcal{P}_0(x)$, is the horizontal curvature of $\mathcal{N}_x\Sigma$ at x.
- The *horizontal Gauss curvature* of Σ at x, is defined as

$$\mathcal{K}_0 = \langle J\vec{n}_0, \nabla_0(\mathcal{P}_0)\rangle - \mathcal{P}_0^2$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Questions/Problems

- Connections of densities to intrinsic notions of rectifiability: e.g. in the sense of Mattila, Serapioni,Serra-Cassano

Questions/Problems

- Connections of densities to intrinsic notions of rectifiability: e.g. in the sense of Mattila, Serapioni,Serra-Cassano
  Note: The t-axis, $T = \{(0,0,t) : t \in \mathbb{R}\}$ is 2-uniform, while not intrinsically rectifiable.

Questions/Problems

- Connections of densities to intrinsic notions of rectifiability: e.g. in the sense of Mattila, Serapioni,Serra-Cassano
  Note: The t-axis, $T = \{(0,0,t) : t \in \mathbb{R}\}$ is 2-uniform, while not intrinsically rectifiable.
- Marstrand's theorem for other metrics?
  e.g. Carnot-Caratheodory metric d_{cc}, or $d_{\mathbb{H}}^{\infty}(p,q) = \|q^{-1} \cdot p\|_{\mathbb{H}}^{\infty}$
  where $\|(z,t)\|_{\mathbb{H}}^{\infty} = \max\{|z|_{\mathbb{R}^{2n}}, |t|^{1/2}\}$

Questions/Problems

- Connections of densities to intrinsic notions of rectifiability: e.g. in the sense of Mattila, Serapioni,Serra-Cassano
  Note: The t-axis, $T = \{(0,0,t) : t \in \mathbb{R}\}$ is 2-uniform, while not intrinsically rectifiable.
- Marstrand's theorem for other metrics?
  e.g. Carnot-Caratheodory metric d_{cc}, or $d_{\mathbb{H}}^{\infty}(p,q) = \|q^{-1} \cdot p\|_{\mathbb{H}}^{\infty}$ where $\|(z,t)\|_{\mathbb{H}}^{\infty} = \max\{|z|_{\mathbb{R}^{2n}}, |t|^{1/2}\}$
- Classification of uniform measures in $(\mathbb{H}^1, d_{\mathbb{H}})$?

Proof of Kirchheim-Preiss theorem in $\mathbb{H}^n$

Let $x_0 \in \operatorname{supp}(\mu)$ and define

$$F(x,s) = \int_{\mathbb{R}^{2n+1}} (\exp(-s\|x^{-1}\cdot z\|_{\mathbb{H}}^4) - \exp(-s\|x_0^{-1}\cdot z\|_{\mathbb{H}}^4))\, d\mu(z),$$

for $x \in \mathbb{H}^n$ and $s > 0$.

1. $F(x,s)$ is well defined (does not depend on x_0)
2. $|F(x,s)| < \infty$ for all $x \in \mathbb{H}^n, s > 0$
3. $F(x,s) = 0$ for all $x \in \operatorname{supp}(\mu)$ and $s > 0$
4. If $x \notin \operatorname{supp}(\mu)$, then $F(x,s) \neq 0$ for some $s > 0$
5. $\operatorname{supp}(\mu) = \bigcap_{s>0}\{x : F(x,s) = 0\}$

Fix $s > 0$. Suffices to show that

$$F_1(x) = \int_{\mathbb{R}^{2n+1}} \exp(-s\|x^{-1}\cdot z\|_{\mathbb{H}}^4)\, d\mu(z)$$

is real analytic.

Proof of Kirchheim-Preiss theorem in $\mathbb{H}^n$

$$F_1(x) = \int_{\mathbb{R}^{2n+1}} \exp(-s\|x^{-1} \cdot z\|_{\mathbb{H}}^4)\, d\mu(z)$$

Define $\tilde{F_1} : \mathbb{C}^{2n+1} \to \mathbb{C}$ for $w = (w_1, \ldots, w_{2n+1}) \in \mathbb{C}^{2n+1}$ as

$$\begin{aligned}\tilde{F_1}(w) = \int_{\mathbb{R}^{2n+1}} \exp(-s\Big[\Big(\sum_{i=1}^{2n}(z_i - w_i)^2\Big)^2 + \Big(z_{2n+1} - w_{2n+1} \\ +2\sum_{i=1}^{n}(w_i z_{i+n} - w_{i+n} z_i)\Big)^2\Big])d\mu(z).\end{aligned}$$

- $|\tilde{F_1}(w)| < \infty$ for all $w \in \mathbb{C}^{2n+1}$
- $\tilde{F_1}|_{\mathbb{R}^{2n+1}} = F_1$
- $\tilde{F_1}$ is holomorphic on $\mathbb{C}^{2n+1}$, so F_1 is real analytic in $\mathbb{R}^{2n+1}$.